

Uncertainty Analysis of Simple Macroeconomic Models Using Angel-Daemon Games

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Abstract

We propose the use of an angel-daemon framework to perform an uncertainty analysis of short-term macroeconomic models with exogenous components. An uncertainty profile \mathcal{U} is a short and macroscopic description of a potentially perturbed situation. The angel-daemon framework uses \mathcal{U} to define a strategic game where two agents, the angel and the daemon, act selfishly having different goals. The Nash equilibria of those games provide the stable strategies in perturbed situations, giving a natural estimation of uncertainty. In this initial work we apply the framework in order to get an uncertainty analysis of linear versions of the IS-LM and the IS-MP models. In those models, by considering uncertainty profiles, we can capture different economical situations. Some of them can be described in terms of macroeconomic policy coordination. In other cases we just analyse the results of the system under some possible perturbation level. Besides providing examples of application we analyse the structure of the Nash equilibria in some particular cases of interest.

Keywords: Uncertainty profiles; strategic games; zero-sum games; angel-daemon games; IS-LM model; IS-MP model

1 Introduction

The distinction between *risk* and *uncertainty* has become increasingly important since (Knight, 1921) discussed it as we have imperfect knowledge of future events in our ever-changing world. Informally, *risk* can be measured by probabilities. In contrast, *uncertainty* refers to something where we cannot even gather the information required to figure out probabilities. However, in practice there is no difference between risk and uncertainty in empirical analysis on the economy and financial markets. Both are measured by historical standard deviation of the variable of interest (Hull, 2010; Arratia, 2014). This paper proposes an alternative to disentangle these seemingly indistinguishable concepts applying ideas from game theory and computer science.

The study of web applications is a field where uncertainty becomes unavoidable. The angel-daemon framework (Gabarro et al., 2014) provides a way to obtain numerical estimates of uncertainty in the execution of a Web service. In such a setting, the uncertainty is captured by an uncertainty profile describing a stressed environment for the execution of the Web application. Uncertainty profiles provide a description of the perceived uncertain behaviour

with respect to possible failing services or execution delays. That is, some sites can potentially misbehave but we are uncertain about the specific sites that will do so. The model attempts to balance positive and negative aspects. Considering only positive aspects (minimizing damage) is usually too optimistic. In the opposite side, being pessimistic (maximizing damage) is also not realistic. Reality often evolves in between optimism and pessimism. To model this situation, the framework considers two agents : *the angel* (\mathfrak{a}), dealing with the optimistic side; and the *daemon* (\mathfrak{d}), dealing with the pessimistic side. These agents act strategically in an associated angel-daemon zero-sum game. In this context, uncertain situations are identified with the Nash equilibria of the angel-daemon game and they are assessed by the value of the game. It is important to emphasize that the results in (Gabarro et al., 2014) are useful to analyse uncertain stable (or timeless) environments. Thus, the framework analyses uncertainty in the short-term and it is not useful for a long-term analysis.

In this paper, we present an angel-daemon ($\mathfrak{a}/\mathfrak{d}$) framework to model uncertainty in short-term macroeconomic models. In the $\mathfrak{a}/\mathfrak{d}$ approach the actions undertaken by \mathfrak{a} and \mathfrak{d} usually go into different directions and can affect the result of underlying systems in unexpected ways. In some cases, \mathfrak{a} or \mathfrak{d} may be identified with policy makers or institutions. In other cases, they may describe a situation created by many interacting agents. We have to be careful about some facts. First, it is usually mostly difficult to picture a policy-maker or an institution acting deliberately as \mathfrak{d} . Second, the distinction $\mathfrak{a}/\mathfrak{d}$ at first sight seems unsuitable when it comes to market participants, as there is not an obvious pessimistic party opposed to an openly optimistic party. Despite the judging names, it is not our intention to associate any moral connotation to them. However, there are actors that might be thought of as performing the role of \mathfrak{a} or the \mathfrak{d} , even with moral issues (Akerlof & Schiller, 2009; Akerlof & Shiller, 2015). We clarify our approach using an example considering the role of the Fed and Wall Street in the Great Recession in terms of $\mathfrak{a}/\mathfrak{d}$. The Fed, as policy-maker, who tried to improve the economic behaviour, is identified with \mathfrak{a} . Wall Street (representing other economic agents) used this policy strategically for their own interest and we identify them with \mathfrak{d} . However, Wall Street (\mathfrak{d}) interacted with Fed (\mathfrak{a}) using a policy in an unexpected way (Greenspan, 2013). It is reasonable to assume that the Great Recession appears as a consequence of the a priori broadly unexpected and strategic interplay between \mathfrak{a} and \mathfrak{d} (Besley & Hennessy, 2009).

Our approach may provide another interpretation of the Keynesian animal spirits (Keynes, 2007). As Keynes pointed out, our decisions to do something are mostly the result of a "spontaneous urge to act" (animal spirits). Thus, in these models, animal spirits (\mathfrak{a} and \mathfrak{d}) can be thought as players in a strategic game. The characters representing \mathfrak{a} and \mathfrak{d} are summarized into the utilities (or dis-utilities) $u_{\mathfrak{a}}$ and $u_{\mathfrak{d}}$. Giving different values to $u_{\mathfrak{a}}$ and $u_{\mathfrak{d}}$ it is possible to shape different situations. For instance, if we set $u_{\mathfrak{a}} = Y$ and $u_{\mathfrak{d}} = T$, the angel tries to maximize the income (in many cases considered as a good issue) and the daemon tries to maximize taxes (considered a bad issue in some schools). By setting $u_{\mathfrak{a}} = -T$ and $u_{\mathfrak{d}} = r$, the angel tries to maximize $-T$ (minimizing T) and \mathfrak{d} tries to maximize the interest rate.

In order to link the $\mathfrak{a}/\mathfrak{d}$ framework with short-term macroeconomic models, we identify the parts of the macroeconomic system to be perturbed. We assume that \mathfrak{a} and \mathfrak{d} can exert some power in the value of some of the exogenous components of the model. We model the power of action by potential perturbations that \mathfrak{a} and \mathfrak{d} might apply to modify the components' estimation. The perturbation values are real numbers, so they can be either positive or negative. We also assume that both agents have limits in the influence they

can exert. This gives rise to an adequate redefinition of uncertainty profiles describing the component's variability in an perturbed situation. As in (Gabarro et al., 2014), once the uncertainty profile is defined, we analyse the stable situations of the corresponding α/\mathfrak{d} -game. The obtained α/\mathfrak{d} games have a richer structure than the initial application as the potential strategic situation cannot always be model by a zero-sum game. Nevertheless, we can define some interesting valuations that can be analysed through zero-sum α/\mathfrak{d} games.

To test the applicability of the α/\mathfrak{d} framework, we start with linear approximation of extensively studied models. In particular, we develop α/\mathfrak{d} analysis of the InvestmentSavings-LiquidityMoney (IS-LM) introduced by (Hicks, 1937, 1980-1981) and the InvestmentSavings-MonetaryPolicy (IS-MP) developed by (Romer, 2000). We have selected those models as they provide different interpretations of the monetary policy, differentiating periods without inflation from those with inflation. Besides defining the framework we model uncertainty of the fiscal policy in the IS-LM model and of the external shocks in the IS-MP model. We either obtain the Nash equilibria or we analyse the properties of Nash equilibria. In particular, we show that if we are uncertain of the fiscal policy, there is always a dominant strategy equilibria.

The paper is structured as follows. In Section 2 we briefly describe the IS-LM and the IS-MP models. Section 3 is devoted to the formulation of the linear approximations to the IS-LM and the IS-MP models describing their exogenous components. In Section 4, we provide a model for the possible perturbations of the set of exogenous components, the so called *perturbation strength model*. Section 5 introduces uncertainty profiles and the associated α/\mathfrak{d} games tailored to the linear IS-LM and IS-MP models and analyses the Nash equilibria of some cases. Section 6 studies the IS-LM model when we are uncertain of the fiscal policy and Section 7 studies the IS-MP model when we are uncertain of the external shocks. Finally, in Section 8, we raise some concluding remarks and discuss some future research.

2 Basic Macroeconomic Models

In order to make this paper self contained and accessible to a wide audience, we describe briefly the two basic Macroeconomic models considered in this paper. The IS-LM which is useful to describe periods with no inflation and the IS-MP in which the inflation is taken explicitly into account by the monetary policy fixed by the Central Bank. Recall that, according to Fisher's equation, the relation between the real interest rate r , the nominal interest rate i and the inflation π is $r = i - \mathbb{E}(\pi)$ where $\mathbb{E}(\pi)$ is the inflationary expectations. When there is no inflation, the real and nominal interest rates coincide.

2.1 IS-LM model

The InvestmentSavings-LiquidityMoney model (IS-LM) (Hicks, 1937) provides a way to express, in equilibrium, the national income and the interest rate as a function of several exogenous components. The IS-LM model describes approximately the monetary market when the gold standard was the norm because the gold can be reasonably represented by the money supply M . Despite its simplicity, this model continues making useful predictions (Krugman, 2011). As we have mention before, when there is no inflation, the nominal and the real interest rate coincide, so only r appears in the equations. This situation of no inflation might happen for long periods, the important fact to remember is that inflation is mostly a twentieth-century phenomenon. Up to World War I, inflation was zero or close to it

(Piketty, 2014). So, for some time, national income Y and interest rate r were the two main macroscopic variables. The IS-LM is described by two equations.

- The IS line $Y = C(Y - T) + I(r) + G$ represents a continuum of equilibria in the goods market. On it, Y is the national income, r is the interest rate. The remaining components are the sum of the annual rates of spending by: the consumers (as a function of the disposable income) $C(Y - T)$; the investors (as a function of the interest rate) $I(r)$ and the government G .
- The LM line $M/P = L(r, Y)$ is interpreted as continuum of equilibrium in the money market. The money supply is M/P , where M is the money and P is the price level. The liquidity preference $L(r, Y)$ is a function of the national income and the interest rate.

An equilibrium point (Y, r) is a solution of the system of equations. These equilibria correspond to the points where both markets are at mutual equilibrium.

2.2 IS-MP Model

In the thirties, the world was in transition from the gold standard and the monetary policy of the central banks changed to deal with inflation. In calm periods, central banks ensure that the money supply grows as economic activity in order to guarantee a low inflation rate of 1 or 2 percent a year. The Central Bank creates new money by lending to banks for very short periods (Piketty, 2014). In Europe, the primary objective of the monetary policy of the European Central Bank is to maintain price stability as a way to contribute to economic growth and job creation. In the pursuit of price stability, the European Central Bank aims at maintaining inflation rates below, but close to, 2% over the medium term (ECB, 2015). The InvestmentSaving-MonetaryPolicy model (IS-MP) (Romer, 2000) considers this new reality about monetary policy and inflation. The MP equation deals with the interest rate as a function of the current inflation, the Central Bank expected inflation and the income gap (Taylor, 1993).

We start with a short description of the *dynamic aggregate demand/aggregate supply model* (dynamic AD/AS model) given in (Mankiw, 2013). The model is dynamic because some variables depend on their lagged (past period) values. In the model t denotes the current period (usually one year) and is given by the following equations.

- The *supply for goods and services* is given by $Y_t = \bar{Y}_t - \alpha(r_t - \rho) + \epsilon_t$. In this equation, the total output for goods and services is Y_t and the economy's natural output is \bar{Y}_t . The parameter $\alpha > 0$ measures the sensitivity of the demand in front of the real interest rate r_t and ρ is the natural rate of interest. The parameter ϵ_t represents the random demand shock.
- The *real interest rate* r_t is given by a simplified version of Fisher's equation with no expectations: $r_t = i_t - \pi_t$. Where i_t is the nominal interest rate and π_t is the inflation rate.
- *Philips curve*. The inflation π_t at period t is described by a version of the Phillips curve as a function of the past inflation π_{t-1} and with no expectations: $\pi_t = \pi_{t-1} + \phi(Y_t - \bar{Y}_t) + v_t$. The parameter $\phi > 0$ measures the responsiveness of the inflation to output fluctuations and v_t is the random supply shock.

Variables	\mathcal{V}
Taxes	$0 < T$
Exogenous government spending	$0 < G$
Money Supply	$0 < M$
Price index	$0 < P$
Parameters	\mathcal{P}
Autonomous consumption	$0 < a$
Marginal propensity to consume	$0 < b < 1$
Exogenous investment	$0 < c$
Interest sensitivity	$0 < d$
Income sensitivity for real money	$0 < e$
Interest sensitivity for real money	$0 < f$

Figure 1: The exogenous components in the linear approximation to the IS-LM model.

- *Monetary policy rule.* It is based on Taylor's Rule (Taylor, 1993). The nominal interest rate i_t is given by $i_t = \pi_t + \rho + \theta_\pi(\pi_t - \pi_t^*) + \theta_Y(Y_t - \bar{Y}_t)$. In this equation, π_t^* is the central bank's target inflation rate and $\theta_\pi > 0$, $\theta_Y > 0$ measure responsiveness.

We avoid temporal dependencies and we consider a version of the IS-MP model assuming that past inflation coincides with the Central Bank target inflation. As before an equilibrium point (Y, π) is a solution of the system of equations.

3 Linear Approximations and Exogenous Components

We introduce here the linear approximations of the IS-LM and the IS-MP models. For both simplified models we make explicit their exogenous components. Let us first fix some notation. We use \mathfrak{M} to denote a linear approximation of a model, informally $\mathfrak{M} \in \{\text{IS-LM}, \text{IS-MP}\}$. For a model \mathfrak{M} , $\mathcal{P}_{\mathfrak{M}}$ denotes the set of exogenous parameters, $\mathcal{V}_{\mathfrak{M}}$ denotes the the set of exogenous variables and $\mathcal{E}_{\mathfrak{M}} = \mathcal{P}_{\mathfrak{M}} \cup \mathcal{V}_{\mathfrak{M}}$ is the set of *exogenous components*. We use set notation like $b \in \mathcal{P}_{\mathfrak{M}}$ or $T \in \mathcal{V}_{\mathfrak{M}}$. When \mathfrak{M} is clear from the context we use \mathcal{E} , \mathcal{P} and \mathcal{V} .

3.1 IS-LM Model

We consider the linear approximation of the IS-LM given in (Baldani et al., 2007).

Definition 1. *The IS-LM model is described by the following equations.*

$$C(Y - T) = a + b(Y - T), \quad I(r) = c - dr, \quad L(r, Y) = eY - fr.$$

The set exogenous components $\mathcal{E} = \mathcal{V} \cup \mathcal{P}$ is given by $\mathcal{V} = \{a, b, c, d, e, f\}$ and $\mathcal{P} = \{T, G, M, P\}$ (see Figure 1).

Let us express the *endogenous variables* $\{Y, r\}$ in equilibrium as a function of \mathcal{E} . The equilibrium condition $Y = Y(r)$ (IS line) gives the equation $Y = a + b(Y - T) + c - dr + G$.

The condition $r = r(Y)$ (MP line) gives $M/P = eY - f r$. Thus, we get

$$Y = \frac{1}{(1-b)}(a + c + G - bT - dr) \quad \text{and} \quad r = \frac{1}{f}(eY - \frac{M}{P}).$$

Using matrix notation, the linear system describing $Y(r)$ and $r(Y)$ can be written as

$$\begin{pmatrix} 1-b & d \\ Pe & -Pf \end{pmatrix} \begin{pmatrix} Y \\ r \end{pmatrix} = \begin{pmatrix} a + c + G - bT \\ M \end{pmatrix}.$$

Solving the system, we get the following expression for the equilibrium point (Y, r) .

$$\begin{pmatrix} Y \\ r \end{pmatrix} = \frac{1}{(1-b)f + de} \begin{pmatrix} f & d/P \\ e & -(1-b)/P \end{pmatrix} \begin{pmatrix} a + c + G - bT \\ M \end{pmatrix}.$$

Defining $g = (1-b)f + de$, we get the following expressions.

$$Y = \frac{f}{g}(a + c + G - bT) + \frac{d}{g} \frac{M}{P} \quad \text{and} \quad r = \frac{e}{g}(a + c + G - bT) - \frac{(1-b)}{g} \frac{M}{P}.$$

As (Y, r) depends on the valuation on \mathcal{E} , when needed we write $(Y(\mathcal{E}), r(\mathcal{E}))$. In order to simplify notation we often use \mathcal{E} to refer also to a valuation of the exogenous components of the model.

Example 3.1. Consider the following valuation of \mathcal{E} :

a	b	c	d	e	f	T	G	M	P
200	3/4	200	25	1	100	100	100	1000	2

The equilibrium point is described by the linear system $Y = 1700 - 100r$, $r = Y/100 - 5$. Solving the system we get $Y = 1100$, $r = 6$. \square

3.2 IS-MP Model

We present now the linear approximation to the IS-MP model from the dynamic AS/AD model as given in (Mankiw, 2013). Recall that the equations on Y_t , r_t and i_t give us the following dynamic aggregate demand line

$$Y_t = \bar{Y}_t - \hat{\alpha}(\pi_t - \pi_t^*) + \hat{\beta}\epsilon_t \quad \text{where} \quad \hat{\alpha} = \frac{\alpha\theta_\pi}{1 + \alpha\theta_Y} \quad \text{and} \quad \hat{\beta} = \frac{1}{1 + \alpha\theta_Y}$$

and the aggregate dynamic aggregate supply line is given by

$$\pi_t = \pi_{t-1} + \phi(Y_t - \bar{Y}_t) + v_t.$$

We consider a simplified linear approximation to the IS-MP in which we avoid dependences of inflation on their lagged values. We consider only the case where the past inflation coincides with the Central Bank target inflation, i.e. $\pi_{t-1} = \pi_t^*$. Furthermore, assuming that period t is known, we consider the equations for a fixed period t so that we can drop the sub-index.

Variables	\mathcal{V}
Central bank's target inflation	π^*
Natural level of output	\bar{Y}
Shock to Y	ϵ
Shock to π	v
Parameters	\mathcal{P}
Y sensitivity to r	$0 < \alpha$
Natural interest rate	$0 < \rho$
π sensitivity to Y in Philips line	$0 < \phi$
i sensitivity to inflation in MP	$0 < \theta_\pi$
i sensitivity to Y in MP	$0 < \theta_Y$

Figure 2: The exogenous components in the linear approximation to the IS-MP model.

Definition 2. *The IS-MP model is described by the following equations.*

$$Y = \bar{Y} - \hat{\alpha}(\pi - \pi^*) + \hat{\beta}\epsilon \quad \text{and} \quad \pi = \pi^* + \phi(Y - \bar{Y}) + v$$

The set of exogenous components $\mathcal{E} = \mathcal{V} \cup \mathcal{P}$ in the IS-MP model is given by $\mathcal{V} = \{\pi^*, \bar{Y}, \epsilon, v\}$ and $\mathcal{P} = \{\alpha, \rho, \phi, \theta_\pi, \theta_Y\}$ (see Figure 2).

From the description of the equations of the IS-LM model we can write the equations of an equilibrium for the endogenous variables $\{Y, \pi\}$. Solving the system, as a function of \mathcal{E} , the equilibrium point is given by

$$Y = \bar{Y} + \hat{\gamma}\epsilon - \hat{\delta}v \quad \text{and} \quad \pi = \pi^* + \hat{\rho}\epsilon + \hat{\mu}v$$

where

$$\begin{aligned} \hat{\gamma} &= \frac{\hat{\beta}}{1 + \hat{\alpha}\phi} = \frac{1}{1 + \alpha(\theta_Y + \phi\theta_\pi)} \quad \text{and} \quad \hat{\delta} = \frac{\hat{\alpha}}{1 + \hat{\alpha}\phi} = \frac{\alpha\theta_\pi}{1 + \alpha(\theta_Y + \phi\theta_\pi)} \\ \hat{\rho} &= \frac{\phi\hat{\beta}}{1 + \hat{\alpha}\phi} = \frac{\phi}{1 + \alpha(\theta_Y + \phi\theta_\pi)} \quad \text{and} \quad \hat{\mu} = \frac{1}{1 + \hat{\alpha}\phi} = \frac{1 + \alpha\theta_Y}{1 + \alpha(\theta_Y + \phi\theta_\pi)}. \end{aligned}$$

The preceding equations can be written in matrix form as

$$\begin{pmatrix} Y \\ \pi \end{pmatrix} = \begin{pmatrix} \hat{\gamma} & -\hat{\delta} \\ \hat{\rho} & \hat{\mu} \end{pmatrix} \begin{pmatrix} \epsilon \\ v \end{pmatrix} + \begin{pmatrix} \bar{Y} \\ \pi^* \end{pmatrix}$$

or, alternatively as

$$\begin{pmatrix} Y \\ \pi \end{pmatrix} = \frac{1}{1 + \alpha(\theta_Y + \phi\theta_\pi)} \begin{pmatrix} 1 & -\alpha\theta_\pi \\ \phi & 1 + \alpha\theta_Y \end{pmatrix} \begin{pmatrix} \epsilon \\ v \end{pmatrix} + \begin{pmatrix} \bar{Y} \\ \pi^* \end{pmatrix}.$$

The equilibrium point is (Y, π) is a function of the valuation \mathcal{E} . As before, when needed we write $(Y(\mathcal{E}), \pi(\mathcal{E}))$.

Example 3.2. Consider the following \mathcal{E} for the IS-MP model:

α	ρ	ϕ	θ_π	θ_Y	π^*	\bar{Y}	ϵ	v
1	2	1/4	1/2	1/2	2	100	1	1/2

We can compute directly

$$\begin{pmatrix} Y \\ \pi \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 8 & -4 \\ 2 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 100 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 1306 \\ 34 \end{pmatrix}$$

Therefore $Y = 1306/13$ and $\pi = 34/13$. □

4 Perturbation Strength Model

Given \mathfrak{M} and a valuation \mathcal{E} of its exogenous components, the computation of some positive aspects might have been underestimated or some negative aspects overestimated. Therefore it makes sense to study \mathfrak{M} under slight (or severe) perturbations. This is achieved studying \mathfrak{M} when the valuation \mathcal{E} is modified strategically. The first component of our proposal is a description of the perceived potential perturbations in the model. Recall that in our approach we want to consider positive and negative aspects by means of the two agents \mathfrak{a} and \mathfrak{d} . These agents act over the model by changing the values of some of the exogenous components inside the limits of our analysis of reality.

Definition 3. Let \mathfrak{M} be a macroeconomic model and let \mathcal{E} be the set of its exogenous components. A perturbation strength model for \mathcal{E} is a set of \mathcal{S} of pairs of real numbers, i.e., $\mathcal{S} = \{(\delta_{\mathfrak{a}}(e), \delta_{\mathfrak{d}}(e)) \mid e \in \mathcal{E}\}$ describing the potential changes that can be applied to the valuations of the exogenous components by \mathfrak{a} and \mathfrak{d} .

Observe that perturbations are real numbers, so they can be either positive or negative. In order to provide intuition on the use of a perturbation strength model let us describe the roles of \mathfrak{a} and \mathfrak{d} through some examples.

Example 4.1. A perturbation strength model \mathcal{S} for the IS-LM Model \mathcal{E} given in Example 3.1 is:

agent	a	b	c, d, e, f	T	G	M	P
\mathfrak{a}	0	+1/20	0	0	+50	0	0
\mathfrak{d}	0	0	0	+50	-25	0	+1

Let us consider the roles of \mathfrak{a} and \mathfrak{d} separately. The angel \mathfrak{a} has the ability to act upon the parameters $\{b, G\}$. The marginal propensity to consume could be increased from 3/4 to 4/5. This is modeled by $\delta_{\mathfrak{a}}(b) = 1/20$. The government spending G might be increased by $\delta_{\mathfrak{a}}(G) = 50$. For any other $e \in \mathcal{E} \setminus \{b, G\}$, \mathfrak{a} has no possibility of acting upon and therefore $\delta_{\mathfrak{a}}(e) = 0$. The daemon \mathfrak{d} has the ability to act upon some of the parameters in $\{P, T, G\}$. The price of goods could increase by $\delta_{\mathfrak{d}}(P) = 1$; taxes could increase $\delta_{\mathfrak{d}}(T) = 50$; spending could decrease by $\delta_{\mathfrak{d}}(G) = -25$. Again, for $e \in \mathcal{E} \setminus \{P, T, G\}$, $\delta_{\mathfrak{d}}(e) = 0$. In this case, who are \mathfrak{a} and \mathfrak{d} ? Are they policy makers? Clearly \mathfrak{a} cannot be a policy maker because a policy maker cannot influence upon the propensity to consume. As we said before, both \mathfrak{a} and \mathfrak{d} are the two faces of the system analyser trying to give values to the possible "perturbations" of the system. The analyser also need to precise if the perturbations will act positively (like \mathfrak{a}) or negatively (like \mathfrak{d}) over the system. □

Example 4.2. In this example \mathfrak{a} and \mathfrak{d} can perturb the predetermined variables \mathcal{V} in the IS-MP model given in Example 3.2. One perturbation strength model \mathcal{S} is

agent	$\alpha, \rho, \phi, \theta_\pi, \theta_Y$	π^*	\bar{Y}	ϵ	v
\mathfrak{a}	0	0	+25	+2	0
\mathfrak{d}	0	+3	0	0	+2

In \mathcal{S} , \mathfrak{a} can potentially act over \bar{Y} and ϵ . That is $\delta_{\mathfrak{a}}(\bar{Y}) = 25$ and $\delta_{\mathfrak{a}}(\epsilon) = 2$ but, for any other $e \in \{\alpha, \rho, \phi, \theta_\pi, \theta_Y, \pi^*, v\}$, $\delta_{\mathfrak{a}}(e) = 0$. \mathfrak{d} can potentially act over π^* and v with $\delta_{\mathfrak{d}}(\pi^*) = 3$ and $\delta_{\mathfrak{d}}(v) = 2$, all other $\delta_{\mathfrak{d}}(e) = 0$. \square

The concrete valuations of \mathcal{E} under perturbation strength model \mathcal{S} depend on the particular selection of components to be perturbed performed by \mathfrak{a} and \mathfrak{d} . For a set $s \subseteq \mathcal{E}$, $\#s$ denotes the number of components in s .

Definition 4. Consider a model \mathfrak{M} having exogenous components \mathcal{E} under a perturbation strength model \mathcal{S} . Given a joint action (a, d) with $a, d \subseteq \mathcal{E}$. The valuation under strength \mathcal{E}' of \mathcal{E} given \mathcal{S} and (a, d) is noted $\text{strength}_{\mathcal{S}}(\mathcal{E})[a, d]$ and it is defined as follows. For any $e \in \mathcal{E}$, $\text{strength}_{\mathcal{S}}(e)[a, d] = e + \delta_{\mathcal{S}}(e)[a, d]$ where

$$\delta_{\mathcal{S}}(e)[a, d] = \begin{cases} 0 & e \notin a \cup d \\ \delta_{\mathfrak{a}}(e) & e \in a \setminus d \\ \delta_{\mathfrak{d}}(e) & e \in d \setminus a \\ \delta_{\mathfrak{a}}(e) + \delta_{\mathfrak{d}}(e) & e \in a \cap d \end{cases}$$

finally, $\text{strength}_{\mathcal{S}}(\mathcal{E})[a, d] = \{\text{strength}_{\mathcal{S}}(e)[a, d] \mid e \in \mathcal{E}\}$.

As we said before, there is a convention in empirical work to measure risk and uncertainty by historical standard deviation. Assume that $e \in \mathcal{E}$ is estimated statistically having expectation μ_e and standard deviation σ_e (Arratia, 2014). The values μ_e and σ_e , can be used to define different perturbation strength models. We focus on different possible cases (even if they are unlikely). Assume that initially we take for the propensity to consume the value $b = \mu_b$. If we are interested in modeling a case where consumption is over the mean μ_e and we feel that this is good (for people), we take \mathfrak{a} as the agent which can potentially increase the estimation of the consumption by setting $\delta_{\mathfrak{a}}(b) = \sigma_b$. In such a case, the modeled system behaves with a propensity to consume $b' = \mu_b + \delta_{\mathfrak{a}}(b) = \mu_e + \sigma_e$. If we like to model the case where consumption is "slightly" under the mean and we feel that this is not so good, we can take $\delta_{\mathfrak{d}}(b) = -\sigma_b/2$ and $b' = \mu_e + \delta_{\mathfrak{d}}(b) = \mu_e - \sigma_b/2$ (assuming $\mu_e > \sigma_b/2$). Finally, we could consider a situation in which both cases are possible together into $b' = \mu_e + \delta_{\mathfrak{a}}(b) + \delta_{\mathfrak{d}}(b) = \mu_e + \sigma_e/2$.

Observe that in the previous case \mathfrak{a} and \mathfrak{d} are far from being policy makers. They just provide a way to focus on possible system behaviours. However, it is also possible to frame this approach into a policies. Consider a policy maker liking to know the possible effects of a tax change. Suppose that he is uncertain about the effects of a tax change $\delta_T > 0$. When the tax increase is considered good, we can take $\delta_{\mathfrak{a}}(T) = \delta_T$ and $\delta_{\mathfrak{d}}(T) = -\delta_T$. In the opposite case (decreasing taxes is a good thing), we can take $\delta_{\mathfrak{a}}(T) = -\delta_T$ and $\delta_{\mathfrak{d}}(T) = \delta_T$.

Finally, a last important case remains to be considered. Imagine an analyser (or a planner) aiming to look at the resilience of a systems under some new and quite open future situation.

Note that this future situation might not be "the one more likely to happen" (according to the planner opinion). It could be a strange, rather unlikely but extremely dangerous (or extremely favourable) situation. Therefore a \mathcal{S} model is a way to study the strength of the system under such situation. Thus, \mathcal{S} it might or might not be derived from a statistical study or a policy plan. It is precisely the task of the analyser to frame the future uncertainty into a perturbation strength model providing the potential actions that \mathfrak{a} and \mathfrak{d} can perform on the parameters.

Let us move to the computation of the equilibrium point in the valuation obtained after a joint action (a, d) of the two agents, denoted as $(Y(\text{strength}_{\mathcal{S}}(\mathcal{E})[a, d]), r(\text{strength}_{\mathcal{S}}(\mathcal{E})[a, d]))$. When \mathcal{S} is clear from the context, and we want to emphasize the role of the choice of parameters (a, d) we note $\text{strength}_{\mathcal{S}}(\mathcal{E})[a, d]$ as $\mathcal{E}(a, d)$ and the equilibrium point as $(Y(a, d), r(a, d))$. The following result points out some basic properties between the different components: \mathcal{E} , \mathcal{S} , (a, d) and strength . In particular, it analyzes explicitly some cases where parts of the system (or the whole system) remains unchanged. In other words, when in \mathcal{S} no unattended modifications can appear.

Lemma 1. *Let \mathfrak{M} be a model having exogenous components \mathcal{E} under a perturbation strength model \mathcal{S} and consider a joint action (a, d) . Then $\text{strength}_{\mathcal{S}}(e)[a, d] = e$ if and only if either $e \notin a \cup d$ or $\delta_{\mathfrak{a}}(e) = \delta_{\mathfrak{d}}(e) = 0$. The whole system remains unperturbed, i.e. $\text{strength}_{\mathcal{S}}(\mathcal{E})[a, d] = \mathcal{E}$ when $\mathcal{S} = \{(0, 0) \mid e \in \mathcal{E}\}$ or $(a, d) = (\emptyset, \emptyset)$.*

Proof. Let (a, d) be a joint action. For $e' \in \mathcal{E}(a, d)$, according to Definition 4, we have $e' = e + \delta_{\mathcal{S}}(e)[a, d]$. Thus, $e' = e$ if and only if $\delta_{\mathcal{S}}(e)[a, d] = 0$ and therefore. The later condition is equivalent to $\delta_{\mathfrak{a}}(e) = \delta_{\mathfrak{d}}(e) = 0$ or $e \notin a \cup d$. The second part of the statement follows trivially from this fact and the definitions. \square

In the following example we provide an application of the Definition 4 to the IS-LM model.

Example 4.3. *We continue with Examples 3.1 and 4.1 under the joint action $(a, d) = (\{b\}, \{P, G\})$. After the joint action we get a new valuation, letting $\mathcal{E}' = \mathcal{E}(\{b\}, \{P, G\})$ and $e' = \text{strength}(e)[\{b\}, \{P, G\}]$.*

We have $\mathcal{E}' = \text{strength}_{\mathcal{S}}(\mathcal{E})[\{b\}, \{P, G\}] = \{a', b', c', d', e', f', T', G', M', P'\}$. The valuations \mathcal{E} and the computation of \mathcal{E}' is sketched in the following table.

agent	choice	a	b	c	d	e	f	T	G	M	P
		200	3/4	200	25	1	100	100	100	1000	2
\mathfrak{a}	$a = \{b\}$		+1/20								
\mathfrak{d}	$d = \{P, G\}$								-25		+1
		a'	b'	c'	d'	e'	f'	T'	G'	M'	P'
		200	4/5	200	25	1	100	100	75	1000	3

Observe that in the joint action \mathfrak{a} acts over the marginal propensity to consume b and \mathfrak{d} acts (at the same time) over the price index P and the exogenous government spending G . As $a \cup d = \{b, P, G\}$, in the tuple $\mathcal{E}' = \mathcal{E}(\{b\}, \{P, G\})$ only the values corresponding to the components b , P and G are perturbed. The other parameters remain unchanged. According to Definition 4 and Lemma 1, $e = e'$ for $e \in \mathcal{E} \setminus \{b, P, G\}$ and $e' = e + \delta_{\mathcal{S}}(e)[a, d]$ for $e \in \{b, P, G\}$.

Taking into account the perturbation strength model \mathcal{S} we get

$$\begin{aligned} b' &= b + \delta_{\mathfrak{a}}(b) = 3/4 + 1/20 = 4/5, \\ P' &= P + \delta_{\mathfrak{d}}(P) = 2 + 1 = 3, \\ G' &= G + \delta_{\mathfrak{d}}(G) = 100 - 25 = 75. \end{aligned}$$

Finally, the equilibrium point corresponding to $\mathcal{E}(\{b\}, \{P, G\})$ is $Y(\{b\}, \{P, G\}) = 28700/27 \approx 1062.96$, $r(\{b\}, \{P, G\}) = 197/27 \approx 7.29$. \square

When \mathfrak{M} is perturbed from valuation \mathcal{E} into \mathcal{E}' by joint action (a, d) , we would like to isolate the effects of the the perturbation by expressing the equilibrium point $(Y(a, d), r(a, d))$ with respect to the non-perturbed equilibrium point (Y, r) . In general this is difficult to obtain, however, when an important part of \mathcal{E} cannot be perturbed, we can take advantage of the linear structure of the IS-LM and IS-MP models and get an explicit expression. In the following lemmas we provide such expressions for some of such cases. Those results will be used later on.

Lemma 2. Consider a perturbation strength model \mathcal{S} for the IS-LM model such that, for $e \in \{b, d, e, f, P\}$, we have $\delta_{\mathfrak{a}}(e) = \delta_{\mathfrak{d}}(e) = 0$. Let (a, d) be a joint action and define

$$\delta_{\mathcal{S}}(a, c, G, T)[a, d] = \delta_{\mathcal{S}}(a)[a, d] + \delta_{\mathcal{S}}(c)[a, d] + \delta_{\mathcal{S}}(G)[a, d] - b \delta_{\mathcal{S}}(T)[a, d].$$

Then, it holds

$$\begin{pmatrix} Y(a, d) \\ r(a, d) \end{pmatrix} = \begin{pmatrix} Y \\ r \end{pmatrix} + \frac{1}{g} \begin{pmatrix} f & d/P \\ e & -(1-b)/P \end{pmatrix} \begin{pmatrix} \delta_{\mathcal{S}}(a, c, G, T)[a, d] \\ \delta_{\mathcal{S}}(M)[a, d] \end{pmatrix}$$

Proof. As \mathfrak{a} and \mathfrak{d} cannot perturb components in $\{b, d, e, f, P\}$, the 2×2 matrix and the value $g = (1-b)f + de$ remain unchanged under \mathcal{S} and any joint action (a, d) . Therefore,

$$\begin{pmatrix} Y(a, d) \\ r(a, d) \end{pmatrix} = \frac{1}{g} \begin{pmatrix} f & d/P \\ e & -(1-b)/P \end{pmatrix} \begin{pmatrix} a' + c' + G' - bT' \\ M' \end{pmatrix}$$

and we get

$$\begin{aligned} a' + c' + G' - bT' &= a + c + G - bT + \delta_{\mathcal{S}}(a)[a, d] + \delta_{\mathcal{S}}(c)[a, d] + \delta_{\mathcal{S}}(G)[a, d] - b \delta_{\mathcal{S}}(T)[a, d] \\ &= a + c + G - bT + \delta_{\mathcal{S}}(a, c, G, T)[a, d], \\ M' &= M + \delta_{\mathcal{S}}(M)[a, d]. \end{aligned}$$

Using straightforward linear algebra the result follows. \square

We are also interested in valuations \mathcal{E} in relation to fiscal policies. In those situations only T and G can suffer a perturbation. In such a case we get the following result.

Lemma 3. Consider a perturbation strength model \mathcal{S} where $\{G, T\}$ are the unique components that can be perturbed. For a joint action (a, d) , we have

$$\begin{pmatrix} Y(a, d) \\ r(a, d) \end{pmatrix} = \begin{pmatrix} Y \\ r \end{pmatrix} + \frac{1}{g} \delta_{\mathcal{S}}(G, T)[a, d] \begin{pmatrix} f \\ e \end{pmatrix}$$

where $\delta_{\mathcal{S}}(G, T)[a, d] = \delta_{\mathcal{S}}(G)[a, d] - b \delta_{\mathcal{S}}(T)[a, d]$.

Proof. As $e \in \{a, b, c, d, e, f, M, P\}$ cannot be perturbed, i.e., $\delta_a(e) = \delta_b(e) = 0$. From Lemma 2, we have $\delta_S(M)[a, d] = 0$. Therefore,

$$\delta_S(a, c, G, T)[a, d] = \delta_S(G)[a, d] - b \delta_S(T)[a, d] = \delta_S(G, T)[a, d]$$

and by computing the matrix product the result follows. \square

Our next result provides the equilibrium point in the IS-MP model when the exogenous parameters cannot be perturbed.

Lemma 4. *Consider a perturbation strength model \mathcal{S} for the IS-MP model such that, for $e \in \mathcal{P}$, $\delta_a(e) = \delta_b(e) = 0$. For any joint action (a, d) it holds that*

$$\begin{pmatrix} Y(a, d) \\ \pi(a, d) \end{pmatrix} = \begin{pmatrix} Y \\ \pi \end{pmatrix} + \begin{pmatrix} \hat{\gamma} & -\hat{\delta} \\ \hat{\rho} & \hat{\mu} \end{pmatrix} \begin{pmatrix} \delta_S(\epsilon)[a, d] \\ \delta_S(v)[a, d] \end{pmatrix} + \begin{pmatrix} \delta_S(\bar{Y})[a, d] \\ \delta_S(\pi^*)[a, d] \end{pmatrix}.$$

Proof. As $\delta_S(e) = 0$, for $e \in \{\alpha, \rho, \phi, \theta_\pi, \theta_Y\}$, $\delta_S(e)[a, d] = e$. Therefore, the 2×2 matrix in the expression for the equilibrium remains unchanged. Thus the values $Y(a, d)$ and $\pi(a, d)$ can be expressed as

$$\begin{pmatrix} Y(a, d) \\ \pi(a, d) \end{pmatrix} = \begin{pmatrix} \hat{\gamma} & -\hat{\delta} \\ \hat{\rho} & \hat{\mu} \end{pmatrix} \begin{pmatrix} \epsilon + \delta_S(\epsilon)[a, d] \\ v + \delta_S(v)[a, d] \end{pmatrix} + \begin{pmatrix} \bar{Y} + \delta_S(\bar{Y})[a, d] \\ \pi^* + \delta_S(\pi^*)[a, d] \end{pmatrix}$$

and, by linearity, the result follows. \square

In Section 7 we will consider the case where only the income Y and the inflation π become uncertain. For such a case we have the following expression for the perturbed equilibrium point.

Lemma 5. *Consider a perturbation strength model \mathcal{S} for the IS-MP model such that, for $e \in \mathcal{E} \setminus \{\bar{Y}, \pi^*\}$, we have $\delta_a(e) = \delta_b(e) = 0$. For any joint action (a, d) , it holds $Y(a, d) = Y + \delta_S(\bar{Y})[a, d]$ and $\pi(a, d) = \pi + \delta_S(\pi^*)[a, d]$.*

Proof. Observe that $\mathcal{E} \setminus \{\bar{Y}, \pi^*\} = \mathcal{P} \cup \{\epsilon, v\}$. As, for $e \in \mathcal{P}$, $\delta_a(e) = \delta_b(e) = 0$ we can use Lemma 4. Thus, we get

$$\begin{pmatrix} Y(a, d) \\ \pi(a, d) \end{pmatrix} = \begin{pmatrix} Y \\ \pi \end{pmatrix} + \begin{pmatrix} \hat{\gamma} & -\hat{\delta} \\ \hat{\rho} & \hat{\mu} \end{pmatrix} \begin{pmatrix} \delta_S(\epsilon)[a, d] \\ \delta_S(v)[a, d] \end{pmatrix} + \begin{pmatrix} \delta_S(\bar{Y})[a, d] \\ \delta_S(\pi^*)[a, d] \end{pmatrix}$$

As $\delta_a(e) = \delta_b(e) = 0$, for any $e \in \{\epsilon, v\}$, we have $\delta_S(\epsilon)[a, d] = \delta_S(v)[a, d] = 0$ and the claimed result follows. \square

We conclude this section with an example illustrating the details of the computation of the equilibrium point in a perturbed scenario.

Example 4.4. *Let us continue with the Example 4.2 in the IS-MP model. Consider the joint action $(a, d) = (\{\epsilon\}, \{\pi^*, v\})$. Writing the new values as $e' = \text{strength}(e)[(\{\epsilon\}, \{\pi^*, v\})]$. As usual,*

$$\mathcal{E}' = \mathcal{E}(\{\epsilon\}, \{\pi^*, v\}) = \{\alpha', \rho', \phi', \theta'_\pi, \theta'_Y, \pi^{*'}, \bar{Y}', \epsilon', v'\}.$$

A sketch of the computation of the perturbed valuation is given in the following table.

agent	choice	α	ρ	ϕ	θ_π	θ_Y	π^*	\bar{Y}	ϵ	v
		1	2	1/4	1/2	1/2	2	100	1	1/2
\mathfrak{a} \mathfrak{d}	$a = \{\epsilon\}$ $d = \{\pi^*, v\}$						+3		+2	+2
		α'	ρ'	ϕ'	θ'_π	θ'_Y	$\pi^{*'}$	\bar{Y}'	ϵ'	v'
		1	2	1/4	1/2	1/2	5	100	3	5/2

As $\epsilon \in a \setminus d$, according to the Definition 4 we have

$$\epsilon' = \text{strength}(\epsilon)[(\{\epsilon\}, \{\pi^*, v\})] = \epsilon + \delta_S[\{\epsilon\}, \{\pi^*, v\}] = \epsilon + \delta_a(\epsilon) = 3.$$

As $\pi \in d \setminus a$, $\pi^{*'} = \pi + \delta_d(\pi^*) = 5$. Similarly $v' = v + \delta_d(v) = 5/2$. All other values remain unchanged. In order to obtain $(Y', \pi^{*'}) = (Y(a, d), \pi(a, d))$, by using Lemma 4, we have

$$\begin{pmatrix} Y' \\ \pi' \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 1306 \\ 34 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 8 & -4 \\ 2 & 12 \end{pmatrix} \begin{pmatrix} \delta_a(\epsilon) \\ \delta_d(v) \end{pmatrix} + \begin{pmatrix} 0 \\ \delta_a(\pi^*) \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 1314 \\ 101 \end{pmatrix}.$$

□

5 Uncertainty Profiles and $\mathfrak{a}/\mathfrak{d}$ games

In this section, we present the model and tools to analyse a situation under a given perturbation strength model. Recall that the perturbation strength model fixes the perturbation to the different exogenous components. Following Gabarro et al. (2014), we introduce *uncertainty profiles* as a tool to provide an *a priori* (global and macroscopic) view of the macroeconomic model under perturbation. An uncertainty profile describes a priori situation where we are uncertain over the specific location where the perturbation will impact the system but we have an approximate idea of the extension of the perturbation. Uncertainty profiles are based in three components. The first one identifies the set of exogenous components that might be perturbed. The second states the limits in the number of components that can suffer perturbation. The third component quantifies (as function of the exogenous components) the benefits for the agents. In such a setting, we are uncertain about the specific subset that will suffer the perturbation.

Definition 5. Let \mathfrak{M} be a macroeconomic model having \mathcal{E} as set of exogenous components. Let S be a perturbation strength model for \mathfrak{M} . An uncertainty profile is a tuple $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u_a, u_d \rangle$ where $\mathcal{A}, \mathcal{D} \subseteq \mathcal{E}$. The spread of the perturbation b_a and b_d , verify $b_a \leq \#\mathcal{A}$ and similarly $b_d \leq \#\mathcal{D}$. The exerted perturbation follows from joint actions (a, d) verifying $a \subseteq \mathcal{A}$, $d \subseteq \mathcal{D}$ with $\#a = b_a$ and $\#d = b_d$. The effects of a joint action are measured by the utility functions u_a and u_d . Given a joint action (a, d) , $u_a(a, d)$ measures \mathfrak{a} 's gain while $u_d(a, d)$ measures \mathfrak{d} 's gain.

The analyser has the perception that, when an angelic component belonging to \mathcal{A} is perturbed, it is unlikely to have a malicious impact. In contrast, when a daemonic component in \mathcal{D} is perturbed, it might well have catastrophic implications. Note that, both \mathcal{A} and \mathcal{D}

determine the potential parameters to be perturbed. In many cases the analyser do not expect that all of them are perturbed "at the same time". Thus, b_a gives the number of components in \mathcal{A} that can be perturbed together. In a similar way b_d gives the limitations for the \mathfrak{d} .

Uncertainty profiles provide a flexible analysis tool. Let us consider some cases of interest. A really pessimistic and global approach like "anything can go wrong" can be modelled taking $\mathcal{A} = \emptyset$ and $\mathcal{D} = \mathcal{E}$. Pessimism increases by selecting bigger values for b_d . The Murphy's law, "anything that can go wrong will go wrong" translates directly as $b_d = \#\mathcal{D}$. When the perturbation of some component can go well or go wrong but not both at the same time, it is enough to assume that $\mathcal{A} \cap \mathcal{D} = \emptyset$. When we are uncertain about a component e and we presume that it could suffer positive and negative influences at the same time, we can take $e \in \mathcal{A} \cap \mathcal{D}$. A completely optimistic perception can be modelled through $\mathcal{A} = \mathcal{E}$, $\mathcal{D} = \emptyset$. Here the degree of optimism is given by the value of b_a . Setting $\mathcal{A} = \mathcal{D} = \mathcal{E}$ models a situation of maximal uncertainty, the different degrees of perturbation, on both sides, are tuned through the values b_a and b_d . For instance, $b_a = \#\mathcal{E}$ and $b_d = \#\mathcal{E}$ forces the \mathfrak{a} and the \mathfrak{d} to act perturbing simultaneously all the components in \mathcal{E} .

The situation described by an uncertainty profile \mathcal{U} is analyzed by means of an associated strategic $\mathfrak{a}/\mathfrak{d}$ game. In such a game \mathfrak{a} and \mathfrak{d} decide their actions strategically. For basics on game theory we refer the reader to (Osborne, 2004; Osborne & Rubinstein, 1994)).

Definition 6. Let \mathfrak{M} be a macroeconomic model having \mathcal{E} as set of exogenous components. Let \mathcal{S} be a perturbation strength model for \mathfrak{M} . Let $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u_a, u_d \rangle$ be an uncertainty profile. The associated angel-daemon strategic game (the $\mathfrak{a}/\mathfrak{d}$ game) $\Gamma(\mathcal{U})$ is defined as $\Gamma(\mathcal{U}) = \langle \{\mathfrak{a}, \mathfrak{d}\}, A_a, A_d, u_a, u_d \rangle$. $\Gamma(\mathcal{U})$ has two players $\{\mathfrak{a}, \mathfrak{d}\}$. The player's actions are $A_a = \{a \subseteq \mathcal{A} \mid \#a = b_a\}$ and $A_d = \{d \subseteq \mathcal{D} \mid \#d = b_d\}$. Their utilities are u_a and u_d .

Notice that, in an $\mathfrak{a}/\mathfrak{d}$ game the set of strategy profiles is $A_a \times A_d$. Thus strategy profiles are permissible joint actions. Recall that, a *pure Nash equilibrium* is a strategy profile such that neither \mathfrak{a} nor \mathfrak{d} can improve the situation by himself (Osborne, 2004). Formally, a strategy profile (a, d) is a pure Nash equilibrium if and only if $u_a(a, d) \geq u_a(a', d)$, for all $a' \in A_a$, and $u_d(a, d) \geq u_d(a, d')$, for all $d' \in A_d$. We note by $\text{PNE}(\Gamma)$ the set of pure Nash equilibria of Γ . When Γ is clear from the context we just write PNE. Given $d \in A_d$ the *best response* of \mathfrak{a} to \mathfrak{d} 's choice d is the set of strategies giving to \mathfrak{a} the maximum utility, i.e., $B_a(d) = \{a \mid u_a(a, d) \geq u_a(a', d) \text{ for all } a' \in A_a\}$. Similarly, $B_d(a) = \{d \mid u_d(a, d) \geq u_d(a, d') \text{ for all } d' \in A_d\}$. It is well known that $(a, d) \in \text{PNE}$ if and only if $a \in B_a(d)$ and $d \in B_d(a)$. Formally, $\text{PNE} = \{(a, d) \mid a \in B_a(d) \text{ and } d \in B_d(a)\}$.

In the following we analyze the existence of PNE for some extreme types of uncertainty profiles.

Lemma 6. Let $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u_a, u_d \rangle$ be an uncertainty profile for \mathfrak{M} . When $b_a = \#\mathcal{A}$ and $b_d = \#\mathcal{D}$, the only PNE of $\Gamma(\mathcal{U})$ is $(\mathcal{A}, \mathcal{D})$. When $b_a = 0$ and $b_d = 0$, the only PNE of $\Gamma(\mathcal{U})$ is (\emptyset, \emptyset) .

Proof. In the first case, the only permissible joint action is $(\mathcal{A}, \mathcal{D})$, In the second case (\emptyset, \emptyset) is the unique permissible joint strategy. Therefore, in both cases, there is only one possible strategy profile and the result follows. \square

The previous lemma considers two extreme cases where neither \mathfrak{a} nor \mathfrak{d} have freedom to make choices. Our next result shows that in an equilibrium, when one of the two agents cannot act, the other agent maximizes his utility.

Lemma 7. Let $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u_a, u_d \rangle$ be an uncertainty profile for \mathfrak{M} . When $b_d = 0$ it holds $\text{PNE}(\Gamma(\mathcal{U})) = \{(a, \emptyset) \mid u_a(a, \emptyset) = \max_{a' \in A_a} u_a(a', \emptyset)\}$. When $b_a = 0$ it holds $\text{PNE}(\Gamma(\mathcal{U})) = \{(\emptyset, d) \mid u_d(\emptyset, d) = \max_{d' \in A_d} u_d(\emptyset, d')\}$.

Proof. In the first case we have $A_a \times A_d = \{(a, \emptyset) \mid a \subseteq \mathcal{A} \text{ and } \#a = b_a\}$. Symmetrically, in the second case we have $A_a \times A_d = \{(\emptyset, d) \mid d \subseteq \mathcal{D} \text{ and } \#d = b_d\}$. The claim follows from the definition of PNE. \square

Let us present some examples of $\mathfrak{a}/\mathfrak{d}$ games arising in the analysis of uncertainty profiles for the IS-LM and the IS-MP models. We have selected them to illustrate some aspects related to the existence of PNE in the associated $\mathfrak{a}/\mathfrak{d}$ game.

Example 5.1. Take $\mathfrak{M} = \text{IS-LM}$, the valuation \mathcal{E} given in Example 3.1 and the perturbation strength model \mathcal{S} introduced in Example 4.1. We will define \mathcal{A} and \mathcal{D} depending on the uncertain scenario we are interested to describe. Consider a situation in which we are interested to know how perturbations affects Y and r when the marginal propensity to consume might be perturbed in an angelic way while the price of goods and taxes might be perturbed in a daemonic way. However, the exogenous government spending might be perturbed in both directions. In consequence we set $\mathcal{A} = \{b, G\}$ and $\mathcal{D} = \{P, G, T\}$. Assume that we do not expect perturbations to be exerted at the same time on more than one component. So, we set $b_a = b_d = 1$. We are interested to know how perturbations affects Y and r . So, we take $u_a = Y$ and we need to precise the interests of \mathfrak{d} with respect to r . In general, a low interest rate r is a mean to achieve higher economic growth and lower unemployment. Now we explore two cases.

First, we want to catch a “worst-case” situation where the perturbation increases the interest rate. The dictum, “having hight interest rate is bad” is captured by setting $u_d = r$. In this case, \mathfrak{d} tries to maximize r . The uncertainty profile $\mathcal{U}_1 = \langle \mathcal{E}, \mathcal{S}, \{b, G\}, \{P, G, T\}, 1, 2, Y, r \rangle$ mimics an uncertain situation asking at the same time for a hight income and a high interest rate. Second, we consider a “best-case” situation. As raising interest rate r has negative effects, we are interested to know what happens when this is not the case. So, r is considered a dis-utility. Raising r is seen as a negative fact. Thus we define $u_d = -r$ (note that maximizing $-r$ is the same as minimizing r). These considerations lead to the uncertainty profile $\mathcal{U}_2 = \langle \mathcal{E}, \mathcal{S}, \{b, G\}, \{P, G, T\}, 1, 2, Y, -r \rangle$.

Let us consider the first case, $\mathcal{U}_1 = \langle \mathcal{E}, \mathcal{S}, \{b, G\}, \{P, G, T\}, 1, 2, Y, r \rangle$ and analyze the PNE of the $\mathfrak{a}/\mathfrak{d}$ game $\Gamma(\mathcal{U}_1)$. According to the definitions, the set of actions for the players are

$$\begin{aligned} A_a &= \{a \subseteq \{b, G\} \mid \#a = 1\} = \{\{b\}, \{G\}\}, \\ A_d &= \{d \subseteq \{P, G, T\} \mid \#d = 2\} = \{\{P, G\}, \{P, T\}, \{T, G\}\} \end{aligned}$$

The utilities are $u_a = Y$ and $u_d = r$ which can be tabulated in the usual bi-matrix form. Recall that a bi-matrix is a table with an entry for each strategy profile holding a pair of values corresponding to the utilities of the two players. Computing the utilities of \mathfrak{a} and \mathfrak{d} (according to the methods developed in Example 4.3), after applying the perturbation to the selected components, we get, for example $Y(\{b\}, \{P, G\}) \approx 1062.96$ and $r(\{b\}, \{P, G\}) \approx 7.29$. The remaining results are summarized in the following bi-matrix representation of $\Gamma(\mathcal{U}_1)$.

		\mathfrak{d}		
		$\{P, G\}$	$\{P, T\}$	$\{T, G\}$
\mathfrak{a}	$\{b\}$	1062.96, 7.29	1029.62, 6.962	1233.33, $22/3 \approx 7.33$
	$\{G\}$	1066.66, $22/3 \approx 7.33$	1041.66, 7.08	1075, 5.75

The sets of \mathfrak{a} 's best responses to \mathfrak{d} 's actions are:

$$B_{\mathfrak{a}}(\{P, G\}) = \{\{G\}\}, B_{\mathfrak{a}}(\{P, T\}) = \{\{G\}\}, B_{\mathfrak{a}}(\{T, G\}) = \{\{b\}\}$$

and the sets of \mathfrak{d} 's best responses to \mathfrak{a} 's actions are:

$$B_{\mathfrak{d}}(\{b\}) = \{\{T, G\}\}, B_{\mathfrak{d}}(\{G\}) = \{\{P, G\}\}$$

The strategy profile $(\{G\}, \{P, G\})$ is a pure Nash equilibrium because $\{G\} \in B_{\mathfrak{a}}(\{P, G\})$ and $\{P, G\} \in B_{\mathfrak{d}}(\{G\})$. Similarly, $(\{b\}, \{T, G\}) \in PNE$. So, we have

$$PNE = \{(\{G\}, \{P, G\}), (\{b\}, \{T, G\})\}$$

The utilities for $(\{G\}, \{P, G\})$ are

$$\begin{aligned} u_{\mathfrak{a}}(\{G\}, \{P, G\}) &= Y(\{G\}, \{P, G\}) = 1066.66 \\ u_{\mathfrak{d}}(\{G\}, \{P, G\}) &= r(\{G\}, \{P, G\}) = 22/3 \approx 7.33 \end{aligned}$$

and the utilities for $(\{b\}, \{T, G\})$ are

$$\begin{aligned} u_{\mathfrak{a}}(\{b\}, \{T, G\}) &= Y(\{b\}, \{T, G\}) = 1233.33 \\ u_{\mathfrak{d}}(\{b\}, \{T, G\}) &= r(\{b\}, \{T, G\}) = 22/3 \approx 7.33 \end{aligned}$$

Thus, $\Gamma(\mathcal{U}_1)$, has more than one PNE. Observe that \mathfrak{a} gets different rewards on these PNE.

In the second case we have $\mathcal{U}_2 = \langle \mathcal{E}, \mathcal{S}, \{b, G\}, \{P, G, T\}, 1, 2, Y, -r \rangle$. The corresponding $\mathfrak{a}/\mathfrak{d}$ game $\Gamma(\mathcal{U}_2)$ is described by the following bi-matrix form.

		\mathfrak{d}		
		$\{P, G\}$	$\{P, T\}$	$\{T, G\}$
\mathfrak{a}	$\{b\}$	1062.96, -7.29	1029.62, -6.962	1233.33, $-22/3 \approx -7.33$
	$\{G\}$	1066.66, $-22/3 \approx -7.33$	1041.66, -7.08	1075, -5.75

As \mathfrak{a} tries to maximize the outcome Y , their sets of best responses are the same as in $\Gamma(\mathcal{U}_1)$. As \mathfrak{d} tries to maximize $-r$ (minimize r), their best responses are $B_{\mathfrak{d}}(\{b\}) = \{\{P, T\}\}$ and $B_{\mathfrak{d}}(\{G\}) = \{\{T, G\}\}$. Observe that in this case $PNE = \emptyset$. As there is no PNE, on any joint strategy, one of the agents (or both), can unilaterally improve his situation. For instance, suppose that initially the system is in $(\{b\}, \{P, G\})$. In this case, \mathfrak{d} has an incentive to change strategy as $B_{\mathfrak{d}}(\{b\}) = \{\{P, T\}\}$. Thus, moving to $(\{b\}, \{P, T\})$. This is denoted as $(\{b\}, \{P, G\}) \xrightarrow{\mathfrak{d}} (\{b\}, \{P, T\})$. Then \mathfrak{a} can improve also his situation. This process give rise to an unstable never-ending behaviour.

$$\begin{aligned} (\{b\}, \{P, G\}) &\xrightarrow{\mathfrak{d}} (\{b\}, \{P, T\}) \xrightarrow{\mathfrak{a}} (\{G\}, \{P, G\}) \xrightarrow{\mathfrak{d}} \\ (\{G\}, \{T, G\}) &\xrightarrow{\mathfrak{a}} (\{b\}, \{T, G\}) \xrightarrow{\mathfrak{d}} (\{b\}, \{P, T\}) \xrightarrow{\mathfrak{a}} \dots \end{aligned}$$

We have considered two cases dealing with opposite interests on r . Many other cases might be of interest, for example when \mathfrak{a} tries to minimize Y and \mathfrak{d} tries to maximize r . In any case, the analyser should transform a perception about uncertainty into one (or several) uncertainty profiles. \square

Example 5.2. Let us continue with the Examples 4.2 and 4.4 of the IS-MP model. We consider the uncertainty profiles where $u_{\mathfrak{a}} = Y$ and analyze the situations in which \mathfrak{d} has opposite interests in the inflation π . We consider again two uncertainty profiles in which the agents have a narrow spread of perturbation: $\mathcal{U}_1 = \langle \mathcal{E}, \mathcal{S}, \{\epsilon, \bar{Y}\}, \{v, \pi^*\}, 1, 1, Y, \pi \rangle$ and $\mathcal{U}_2 = \langle \mathcal{E}, \mathcal{S}, \{\epsilon, \bar{Y}\}, \{v, \pi^*\}, 1, 1, Y, -\pi \rangle$.

Computing the utilities as in Example 4.4, for instance $Y(\{\epsilon\}, \{v\}) = 13014/13$ and $\pi(\{\epsilon\}, \{v\}) = 62/13$, we get the following bi-matrix form for $\Gamma(\mathcal{U}_1)$.

		\mathfrak{d}	
		$\{v\}$	$\{\pi^*\}$
\mathfrak{a}	$\{\epsilon\}$	1314/13, 62/13	1322/13, 77/13
	$\{\bar{Y}\}$	1623/13, 58/13	1631/13, 73/13

As before, we compute the sets of best responses: $B_{\mathfrak{a}}(\{v\}) = B_{\mathfrak{a}}(\{\pi^*\}) = \{\{\bar{Y}\}\}$ and $B_{\mathfrak{d}}(\{\epsilon\}) = B_{\mathfrak{d}}(\{\bar{Y}\}) = \{\{\pi^*\}\}$. Observe that the game has a unique pure Nash equilibrium. Thus, $\text{PNE} = \{(\{\bar{Y}\}, \{\pi^*\})\}$.

For $\mathcal{U}_2 = \langle \mathcal{E}, \mathcal{S}, \{\epsilon, \bar{Y}\}, \{v, \pi^*\}, 1, 1, Y, -\pi \rangle$ the $\mathfrak{a}/\mathfrak{d}$ game $\Gamma(\mathcal{U}_2)$ is described as follows.

		\mathfrak{d}	
		$\{v\}$	$\{\pi^*\}$
\mathfrak{a}	$\{\epsilon\}$	1314/13, -62/13	1322/13, -77/13
	$\{\bar{Y}\}$	1623/13, -58/13	1631/13, -73/13

The sets of \mathfrak{d} 's best responses are $B_{\mathfrak{d}}(\{\epsilon\}) = B_{\mathfrak{d}}(\{\bar{Y}\}) = \{\{v\}\}$. So, $\text{PNE} = \{(\{\bar{Y}\}, \{v\})\}$. \square

In view of the previous examples an important question is whether we can characterize uncertainty profiles with respect to the existence of PNE in the associated $\mathfrak{a}/\mathfrak{d}$ game. In the following section we provide such an answer for particular cases generalizing the situations presented in the previous examples. Now we turn our attention to the more general notion of *mixed Nash equilibrium*.

It is well known that although a strategic game might not have a PNE, it always has a Nash equilibrium on mixed strategies (Osborne & Rubinstein, 1994). So, from the $\mathfrak{a}/\mathfrak{d}$ game, the stable perturbed situations are described by the mixed strategies of the players in the Nash equilibria. A *mixed strategy* for a player is a probability distribution on its set of actions. Thus in an $\mathfrak{a}/\mathfrak{d}$ game, a *mixed strategy profile* is a tuple (α, β) where $\alpha : A_{\mathfrak{a}} \rightarrow [0, 1]$ and $\beta : A_{\mathfrak{d}} \rightarrow [0, 1]$ are probability distributions. The utility for player $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$ of a mixed strategy profile (α, β) is defined as

$$u_{\mathfrak{p}}(\alpha, \beta) = \sum_{(a, d) \in A_{\mathfrak{a}} \times A_{\mathfrak{d}}} \alpha(a) \beta(d) u_{\mathfrak{p}}(a, d)$$

The following example illustrates the notion of mixed strategy.

Example 5.3. We provide an example of mixed strategies, for the game $\Gamma(\mathcal{U}_1)$ defined in Example 5.1. Let (α, β) a mixed strategy for $\Gamma(\mathcal{U}_1)$.

As $A_a = \{\{b\}, \{G\}\}$ we can describe α as a function that assigns probability x to $\{b\}$ and probability $1 - x$ to $\{G\}$, for some $0 \leq x \leq 1$. We represent such a function as

$$\alpha = (\alpha(\{b\}), \alpha(\{G\})) = (x, 1 - x) \text{ with } 0 \leq x \leq 1$$

As $A_d = \{\{P, G\}, \{P, T\}, \{T, G\}\}$ we can describe a probability distribution on A_d as

$$\begin{aligned} \beta &= (\beta(\{P, G\}), \beta(\{P, T\}), \beta(\{T, G\})) \\ &= (y, z, 1 - y - z) \text{ with } 0 \leq y, z \leq 1 \text{ and } y + z \leq 1 \end{aligned}$$

We can represent a generic mixed strategy profile using the bi-matrix form of $\Gamma(\mathcal{U}_1)$ and adding the probabilities of each action.

		\mathfrak{d}			
		$\{P, G\}$	$\{P, T\}$	$\{T, G\}$	
		y	z	$1 - y - z$	
\mathfrak{a}	$\{b\}$	x	1062.96, 7.29	1029.62, 6.962	1233.33, $22/3 \approx 7.33$
	$\{G\}$	$1 - x$	1066.66, $22/3 \approx 7.33$	1041.66, 7.08	1075, 5.75

The utility, for $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$, is computed as

$$\begin{aligned} u_{\mathfrak{p}}(\alpha, \beta) &= \alpha(\{b\})\beta(\{P, G\})u_{\mathfrak{p}}(\{b\}, \{P, G\}) + \dots \\ &\quad \dots + \alpha(\{G\})\beta(\{T, G\})u_{\mathfrak{p}}(\{G\}, \{T, G\}) \end{aligned}$$

In particular setting $x = 1/3$, $y = 1/4$ and $z = 1/2$. The strategy profile is $(\alpha, \beta) = ((1/3, 2/3), (1/4, 1/2, 1/4))$ and the utilities for the two players are

$$\begin{aligned} u_a(\alpha, \beta) &= \frac{1}{3} \times \frac{1}{4} \times 1062.96 + \dots = 1067.124 \\ u_d(\alpha, \beta) &= \frac{1}{3} \times \frac{1}{4} \times 7.29 + \dots = 6.9195 \end{aligned}$$

□

Let Δ_a and Δ_d denote the set of mixed strategies for players \mathfrak{a} and \mathfrak{d} , respectively. A pure strategy profile (a, d) is a special case of a mixed strategy profile (α, β) in which $\alpha(a) = 1$ and $\beta(d) = 1$. The definition of Nash equilibrium extends the conditions for pure strategies to mixed strategies. We adapt to $\mathfrak{a}/\mathfrak{d}$ games the characterization of a mixed Nash equilibrium given in (Osborne, 2004) that we will use to prove that some mixed strategies are Nash equilibria.

Property 8. A mixed strategy profile (α, β) is a mixed Nash equilibrium in $\Gamma(\mathcal{U})$ if the following two symmetric conditions hold. For all $a \in A_a$, when $\alpha(a) > 0$ we have $u_a(\alpha, \beta) = u_a(a, \beta)$, otherwise $u_a(\alpha, \beta) \geq u_a(a, \beta)$. For all $d \in A_d$, when $\beta(d) > 0$ we have $u_d(\alpha, \beta) = u_d(\alpha, d)$, otherwise $u_d(\alpha, \beta) \geq u_d(\alpha, d)$.

In the following example we use this characterization to obtain a mixed Nash equilibrium.

Example 5.4. Let us continue with the Example 5.1. We have seen that the \mathbf{a}/\mathbf{d} game corresponding to $\mathcal{U}_2 = \langle \mathcal{E}, \mathcal{S}, \{b, G\}, \{P, G, T\}, 1, 2, Y, -r \rangle$ has no PNE. Let us compute a mixed Nash equilibrium. An strategy $\alpha \in \Delta_{\mathbf{a}}$ can be described as $\alpha = (\alpha(\{b\}), \alpha(\{G\})) = (x, 1 - x)$ with $0 \leq x \leq 1$. As \mathbf{d} is rational, \mathbf{d} will never choose $\{P, G\}$ because, independently of \mathbf{a} 's action, $\{P, T\}$ or $\{T, G\}$ provide better utility. Therefore, according to Property 8, $\beta(\{P, G\}) = 0$ in any Nash equilibria (α, β) . Therefore, we can restrict our search to $\beta \in \Delta_{\mathbf{d}}$ having the form

$$\beta = (\beta(\{P, G\}), \beta(\{P, T\}), \beta(\{T, G\})) = (0, y, 1 - y),$$

for $0 \leq y \leq 1$. Using property 8 we can express the \mathbf{d} utilities and the conditions for such an (α, β) to be a mixed Nash equilibrium.

$$\begin{aligned} u_{\mathbf{d}}(\alpha, \{P, G\}) &= -x \times 7.29 - (1 - x) \times \frac{22}{3} = 0.04333333 \times x - 7.333333 \\ u_{\mathbf{d}}(\alpha, \{P, T\}) &= -x \times 6.962 - (1 - x) \times 7.08 = 0.118 \times x - 7.08 \\ u_{\mathbf{d}}(\alpha, \{T, G\}) &= -x \times \frac{22}{3} - (1 - x) \times 5.75 = -1.583333 \times x - 5.75 \\ u_{\mathbf{d}}(\alpha, \{P, G\}) &\leq u_{\mathbf{d}}(\alpha, \{P, T\}) = u_{\mathbf{d}}(\alpha, \{T, G\}) \end{aligned}$$

The equality $u_{\mathbf{d}}(\alpha, \{P, T\}) = u_{\mathbf{d}}(\alpha, \{T, G\})$ determines $\alpha = (0.78174, 0.21826)$

To obtain the value of y we state the utilities and conditions for \mathbf{a} .

$$\begin{aligned} u_{\mathbf{a}}(\{b\}, \beta) &= y \times 1029.62 + (1 - y) \times 1233.33 = -203.71 \times y + 1233.33 \\ u_{\mathbf{a}}(\{G\}, \beta) &= y \times 1041.66 + (1 - y) \times 1075 = -33.34 \times y + 1075 \end{aligned}$$

To get a Nash equilibrium we need $u_{\mathbf{a}}(\{b\}, \beta) = u_{\mathbf{a}}(\{G\}, \beta)$ then

$$\beta = (0, 0.9293303, 0.07066972).$$

Therefore, $(\alpha, \beta) = ((0.78174, 0.21826), (0, 0.9293303, 0.07066972))$ is a mixed Nash equilibrium for $\Gamma(\mathcal{U}_2)$. Finally, we can compute the players' utilities in such an equilibrium:

$$\begin{aligned} u_{\mathbf{a}}(\alpha, \beta) &= x \times y \times 1029.62 + x \times (1 - y) \times 1233.33 + (1 - x) \times y \times 1041.66 \\ &\quad + (1 - x) \times (1 - y) \times 1075 = 1044.016 \\ u_{\mathbf{d}}(\alpha, \beta) &= -x \times y \times 6.962 - x \times (1 - y) \times (22/3) - (1 - x) \times y \times 7.08 \\ &\quad - (1 - x) \times (1 - y) \times 5.75 = -6.98775 \end{aligned}$$

□

As we have seen in the previous examples games can have more than one Nash equilibrium and get different utilities in those situations. This property does not happen in the case of two players *zero-sum games* (von Neumann & Morgenstern, 1944). In a zero-sum game, the sum of the utilities of both players is always zero. A zero-sum game can have several Nash equilibria but all of them provide the same utility for the players. As all the Nash equilibria have the same utility for the first player, this utility is called the *value of the game*.

To obtain a zero-sum \mathbf{a}/\mathbf{d} game, we need uncertainty profiles where $u_{\mathbf{d}} + u_{\mathbf{a}} = 0$. We can model this situation by considering a unique objective function u so that $u_{\mathbf{a}} = u$ and $u_{\mathbf{d}} = -u$. In such a case the angelic and daemonic interests go exactly in opposite directions. In a zero-sum situation we simplify notation. In an uncertainty profile we introduce only the

function u : $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u \rangle$. The description of the corresponding zero-sum \mathbf{a}/\mathbf{d} is done by providing the table with the u values, replacing the usual bi-matrix form. The value of the game $\Gamma(\mathcal{U})$ is denoted by $\nu(\mathcal{U})$ and it can be expressed as

$$\nu(\mathcal{U}) = \min_{\alpha \in \Delta_{\mathbf{a}}} \max_{\beta \in \Delta_{\mathbf{d}}} u(\alpha, \beta) = \max_{\beta \in \Delta_{\mathbf{d}}} \min_{\alpha \in \Delta_{\mathbf{a}}} u(\alpha, \beta)$$

One natural way to get a zero-sum situation associated to a given uncertainty profile $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u_a, u_d \rangle$, is to consider a strategy profile (a, d) as a perturbation of the strategy profile (\emptyset, \emptyset) . One way of doing so is to consider the uncertainty profile $\mathcal{U}' = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u \rangle$ where

$$u(a, d) = u_a(a, d) - \frac{u_a(\emptyset, \emptyset)}{u_d(\emptyset, \emptyset)} u_d(a, d).$$

In the following examples we analyze the values of such games derived from uncertainty profiles on the IS-LM and the IS-MP models considered before.

Example 5.5. Consider the uncertainty profile $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \{b, G\}, \{P, G, T\}, 1, 2, u \rangle$ with $u(a, d) = Y(a, d) - \frac{Y(\emptyset, \emptyset)}{r(\emptyset, \emptyset)} r(a, d)$ for the IS-LM model derived from the uncertainty profile considered in Example 3.1. Recall that the system's solution is $(Y, r) = (1100, 6)$. According to Lemma 1, when $(a, d) = (\emptyset, \emptyset)$ it holds $(Y(\emptyset, \emptyset), r(\emptyset, \emptyset)) = (Y, r) = (1100, 6)$ therefore $u(a, d) = Y(a, d) - \frac{1100}{6} r(a, d)$. Simplifying $u(a, d) = Y(a, d) - \frac{550}{3} r(a, d)$. The corresponding zero-sum \mathbf{a}/\mathbf{d} game $\Gamma(\mathcal{U})$ is described by the table of u :

		\mathbf{d}		
		$\{P, G\}$	$\{P, T\}$	$\{T, G\}$
\mathbf{a}	$\{b\}$	$-22250/81 \approx -274.69$	$-20000/81 \approx -246.91$	$-1000/9 \approx -111.11$
	$\{G\}$	$-2500/9 \approx -277.77$	$-4625/18 \approx -256.94$	$125/6 \approx 20.83$

Observe that there is one PNE at $(\{b\}, \{P, G\})$, therefore we know that all NE will provide the same utility and that $\nu(\mathcal{U}) = -22250/81$. \square

Example 5.6. Let us reconsider the Example 5.2. This leads to the zero-sum e uncertainty profile $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \{\epsilon, \bar{Y}\}, \mathcal{D} = \{v, \pi^*\}, 1, 1, u \rangle$ where $u(a, d) = Y(a, d) - (Y(\emptyset, \emptyset)/\pi(\emptyset, \emptyset))\pi(a, d)$. According to Example 3.2, $Y(\emptyset, \emptyset) = 1306/13$ and $\pi(\emptyset, \emptyset) = 34/13$. In this case $\Gamma(\mathcal{U})$ is described by the following table.

		\mathbf{d}	
		$\{v\}$	$\{\pi^*\}$
\mathbf{a}	$\{\epsilon\}$	$-1396/17 \approx -82.11$	$-2139/17 \approx -125.82$
	$\{\bar{Y}\}$	$-791/17 \approx -46.52$	$-1534/17 \approx -90.23$

The sets of best responses are $B_{\mathbf{a}}(\{v\}) = B_{\mathbf{a}}(\{\pi^*\}) = \{\{\bar{Y}\}\}$ and $B_{\mathbf{d}}(\{\epsilon\}) = B_{\mathbf{d}}(\{\bar{Y}\}) = \{\{\pi^*\}\}$. There is a PNE $(\{\bar{Y}\}, \{\pi^*\})$ and therefore the value of the game is $\nu(\mathcal{U}) = -1534/17 \approx -90.23$. \square

6 Uncertainty in the IS-LM Model

We consider the case in which a perturbation is exerted only on the fiscal policy, i.e., $\mathcal{A} = \mathcal{D} = \{G, T\}$. Furthermore, we assume also that \mathfrak{a} and \mathfrak{d} can control just one of the components. We analyze two situations with respect to the utilities. In the first case \mathfrak{a} 's objective is to increase the income as much as possible while \mathfrak{d} tries to increase the interest rate. In the second case we consider a zero-sum approach taking $u = Y - kr$.

6.1 A Case of Fiscal Policy under Uncertainty in the IS-LM Model

Consider the case where \mathfrak{a} and \mathfrak{d} have the capability to act over T and G , that is $A_{\mathfrak{a}} = A_{\mathfrak{d}} = \{\{T\}, \{G\}\}$ and $u_{\mathfrak{a}} = Y$ and $u_{\mathfrak{d}} = r$ (as it was the case in Example 5.1). We ask if the addition of uncertainty can generate a situation in which no PNE exists. We provide a negative answer. Before stating the result. Let us remind, in the context of $\mathfrak{a}/\mathfrak{d}$ games, the definition of *dominant strategy equilibrium* (DSE) taken from (Osborne & Rubinstein, 1994). A DSE is a joint action (a, d) such that, for any other joint action $(a', d') \in A_{\mathfrak{a}} \times A_{\mathfrak{d}}$, $u_{\mathfrak{a}}(a, d') \geq u_{\mathfrak{a}}(a', d')$ and $u_{\mathfrak{d}}(a', d) \geq u_{\mathfrak{d}}(a', d')$. As pointed out in (Osborne & Rubinstein, 1994), in a DSE, the action of *every* player is a best response independently on which are the actions taken by the other players. Our next result proves the existence of DSE when the perturbation has to be exerted only in one component

Theorem 9. *Let \mathcal{S} be a perturbation strength model, for the IS-LM model, given by*

agent	a, b, c, d, e, f	T	G	M, P
\mathfrak{a}	0	$\delta_{\mathfrak{a}}(T)$	$\delta_{\mathfrak{a}}(G)$	0
\mathfrak{d}	0	$\delta_{\mathfrak{d}}(T)$	$\delta_{\mathfrak{d}}(G)$	0

For the uncertainty profile $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \{G, T\}, \{G, T\}, 1, 1, Y, r \rangle$, the associated $\mathfrak{a}/\mathfrak{d}$ game $\Gamma(\mathcal{U})$ in bi-matrix form is given by

		\mathfrak{d}	
		$\{T\}$	$\{G\}$
\mathfrak{a}	$\{T\}$	$u_{\mathfrak{a}} = Y - \frac{f}{g}b(\delta_{\mathfrak{a}}(T) + \delta_{\mathfrak{d}}(T))$ $u_{\mathfrak{d}} = r - \frac{e}{g}b(\delta_{\mathfrak{a}}(T) + \delta_{\mathfrak{d}}(T))$	$u_{\mathfrak{a}} = Y + \frac{f}{g}(\delta_{\mathfrak{d}}(G) - b\delta_{\mathfrak{a}}(T))$ $u_{\mathfrak{d}} = r + \frac{e}{g}(\delta_{\mathfrak{d}}(G) - b\delta_{\mathfrak{a}}(T))$
	$\{G\}$	$u_{\mathfrak{a}} = Y + \frac{f}{g}(\delta_{\mathfrak{a}}(G) - b\delta_{\mathfrak{d}}(T))$ $u_{\mathfrak{d}} = r + \frac{e}{g}(\delta_{\mathfrak{a}}(G) - b\delta_{\mathfrak{d}}(T))$	$u_{\mathfrak{a}} = Y + \frac{f}{g}(\delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G))$ $u_{\mathfrak{d}} = r + \frac{e}{g}(\delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G))$

Furthermore, it holds that $\Gamma(\mathcal{U})$ has always a DSE.

Proof. Observe that, in \mathcal{S} , $\delta_{\mathfrak{a}}(x) = \delta_{\mathfrak{d}}(x) = 0$, for $x \notin \{G, T\}$. Furthermore, in $\Gamma(\mathcal{U})$ as $\mathcal{A} = \{T, G\}$ but $b_{\mathfrak{a}} = 1$, the angel \mathfrak{a} has only the possibility to act over one parameter therefore $A_{\mathfrak{a}} = \{\{T\}, \{G\}\}$ and similarly for \mathfrak{d} . Thus, in $\Gamma(\mathcal{U})$ the sets of actions are $A_{\mathfrak{a}} = A_{\mathfrak{d}} = \{\{T\}, \{G\}\}$. As $u_{\mathfrak{a}}(a, d) = Y(a, d)$ and $u_{\mathfrak{d}}(a, d) = r(a, d)$ the $\mathfrak{a}/\mathfrak{d}$ game is

		\mathfrak{d}	
		$\{T\}$	$\{G\}$
\mathfrak{a}	$\{T\}$	$Y(\{T\}, \{T\}), r(\{T\}, \{T\})$	$Y(\{T\}, \{G\}), r(\{T\}, \{G\})$
	$\{G\}$	$Y(\{G\}, \{T\}), r(\{G\}, \{T\})$	$Y(\{G\}, \{G\}), r(\{G\}, \{G\})$

According to Lemma 3

$$Y(a, d) = Y + \frac{f}{g}\delta_S(G, T)[a, d], \quad r(a, d) = r + \frac{e}{g}\delta_S(G, T)[a, d].$$

To find the values $Y(a, d)$ and $r(a, d)$ we need to compute the values of $\delta_S(G, T)[a, d] = \delta_S(G)[a, d] - b\delta_S(T)[a, d]$. For instance, when $(a, d) = (\{T\}, \{T\})$, according to Definition 4, $\delta_S(G)[\{T\}, \{T\}] = 0$, $\delta_S(T)[\{T\}, \{T\}] = \delta_a(T) + \delta_d(T)$ and $\delta_S(G, T)[\{T\}, \{T\}] = -b(\delta_a(T) + \delta_d(T))$. The other cases are $\delta_S(G, T)[\{T\}, \{G\}] = \delta_d(G) - b\delta_a(T)$, $\delta_S(G, T)[\{G\}, \{T\}] = \delta_a(G) - b\delta_d(T)$ and $\delta_S(G, T)[\{G\}, \{G\}] = \delta_a(G) + \delta_d(G)$. From those computations we conclude that the bi-matrix form of the $\mathfrak{a}/\mathfrak{d}$ corresponds to the statement.

In order to analyse the structure of Nash equilibria, let us study best responses by case analysis. Consider \mathfrak{a} 's utility increment when \mathfrak{d} is in $\{T\}$ and \mathfrak{a} moves from $\{T\}$ to $\{G\}$ written as $T \rightarrow G$.

$$\begin{aligned} u_a(\{G\}, \{T\}) - u_a(\{T\}, \{T\}) &= Y(\{G\}, \{T\}) - Y(\{T\}, \{T\}) \\ &= (f/g)(\delta_S(G, T)[\{G\}, \{T\}] - \delta_S(G, T)[\{T\}, \{T\}]) \\ &= (f/g)(b\delta_a(T) + \delta_a(G)) \end{aligned}$$

Defining $\mu_{a, T \rightarrow G} = b\delta_a(T) + \delta_a(G)$ we write $u_a(\{G\}, \{T\}) - u_a(\{T\}, \{T\}) = (f/g)\mu_{a, T \rightarrow G}$. We see that, abstracting from factor f/g , the expression $\mu_{a, T \rightarrow G}$ measures the increment (in particular the sign) of \mathfrak{a} 's utility. Due to the linear structure of the model, $\mu_{d, T \rightarrow G} = b\delta_d(T) + \delta_d(G)$ also makes sense to measure \mathfrak{d} 's increment. We prove the following claim: given $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$ and defining $\mu_{\mathfrak{p}, T \rightarrow G} = b\delta_{\mathfrak{p}}(T) + \delta_{\mathfrak{p}}(G)$ it holds that

$$B_{\mathfrak{p}}(\{T\}) = B_{\mathfrak{p}}(\{G\}) = \begin{cases} \{G\} & \text{if } \mu_{\mathfrak{p}, T \rightarrow G} > 0 \\ \{T, G\} & \text{if } \mu_{\mathfrak{p}, T \rightarrow G} = 0 \\ \{T\} & \text{if } \mu_{\mathfrak{p}, T \rightarrow G} < 0 \end{cases}$$

To prove the claim, we start with \mathfrak{a} and consider $B_a(\{T\})$ and $B_a(\{G\})$ separately. In order to find $B_a(\{T\})$, we know the change when \mathfrak{a} moves between $\{T\}$ and $\{G\}$, but $Y(\{G\}, \{T\}) - Y(\{T\}, \{T\}) = (f/g)\mu_{a, T \rightarrow G}$. For $B_a(\{G\})$, we have also $Y(\{G\}, \{G\}) - Y(\{T\}, \{G\}) = (f/g)\mu_{a, T \rightarrow G}$. Therefore, for $d \in A_{\mathfrak{d}}$, we have $Y(\{G\}, d) - Y(\{T\}, d) = (f/g)\mu_{a, T \rightarrow G}$. As $f/g > 0$, we conclude that

$$B_a(\{T\}) = B_a(\{G\}) = \begin{cases} \{G\} & \text{if } \mu_{a, T \rightarrow G} > 0 \\ \{T, G\} & \text{if } \mu_{a, T \rightarrow G} = 0 \\ \{T\} & \text{if } \mu_{a, T \rightarrow G} < 0 \end{cases}$$

For \mathfrak{d} , we have $r(a, \{G\}) - r(a, \{T\}) = (e/g)\mu_{d, T \rightarrow G}$ for $a \in A_{\mathfrak{a}}$. As $e/g > 0$, we obtain

$$B_{\mathfrak{d}}(\{T\}) = B_{\mathfrak{d}}(\{G\}) = \begin{cases} \{G\} & \text{if } \mu_{d, T \rightarrow G} > 0 \\ \{T, G\} & \text{if } \mu_{d, T \rightarrow G} = 0 \\ \{T\} & \text{if } \mu_{d, T \rightarrow G} < 0 \end{cases}$$

As \mathfrak{a} and \mathfrak{d} have similar structure, taking any agent $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$, the claim follows.

According to the best responses, by case analysis, we find the following PNE structure:

	$\mu_{\mathfrak{a}, T \rightarrow G} > 0$	$\mu_{\mathfrak{a}, T \rightarrow G} = 0$	$\mu_{\mathfrak{a}, T \rightarrow G} < 0$
$\mu_{\mathfrak{d}, T \rightarrow G} > 0$	$\{G, G\}$	$\{G, T\}, \{G, G\}$	$\{G, T\}$
$\mu_{\mathfrak{d}, T \rightarrow G} = 0$	$\{T, G\} \{G, G\}$	$\{G, G\} \{G, T\} \{T, G\} \{T, T\}$	$\{T, T\} \{G, T\}$
$\mu_{\mathfrak{d}, T \rightarrow G} < 0$	$\{T, G\}$	$\{T, T\} \{T, G\}$	$\{T, T\}$

To prove the existence of a DSE we consider first the case $\mu_{\mathfrak{p}, T \rightarrow G} > 0$, for $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$. Let us show that, in such a case, $(\{G\}, \{G\})$ is a dominant strategy equilibrium. When $\mathfrak{p} = \mathfrak{a}$ the conditions for DSE are $u_{\mathfrak{a}}(\{G\}, d) \geq u_{\mathfrak{a}}(\{T\}, d)$ for $d \in A_{\mathfrak{d}}$. As for $d \in A_{\mathfrak{d}}$

$$u_{\mathfrak{a}}(\{G\}, d) - u_{\mathfrak{a}}(\{T\}, d) = (f/g)\mu_{\mathfrak{a}, T \rightarrow G} > 0$$

the constraint hold. When $\mathfrak{p} = \mathfrak{d}$ the condition is $u_{\mathfrak{d}}(a, \{G\}) \geq u_{\mathfrak{d}}(a, \{T\})$ for any $a \in A_{\mathfrak{a}}$. As

$$u_{\mathfrak{d}}(a, \{G\}) - u_{\mathfrak{d}}(a, \{T\}) = (e/g)\mu_{\mathfrak{d}, T \rightarrow G} > 0$$

constraints hold and $(\{G\}, \{G\})$ is a dominant strategy equilibrium when $\mu_{\mathfrak{a}, T \rightarrow G} > 0$. The remaining cases follow by a similar argument. \square

In the situation analyzed on Theorem 9 there is the possibility of modifying government spending or taxes. For both components there are positive and negative aspects, this is model by the perturbation strength parameters $\delta_{\mathfrak{a}}(G)$, $\delta_{\mathfrak{d}}(G)$, $\delta_{\mathfrak{a}}(T)$, and $\delta_{\mathfrak{d}}(T)$. If we take into account *all* the possible perturbed actions together we get that

$$\hat{G} = G + \delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G), \quad \hat{T} = T + \delta_{\mathfrak{a}}(T) + \delta_{\mathfrak{d}}(T).$$

In such a case, the income \hat{Y} and the interest rate \hat{r} are given by

$$\begin{pmatrix} \hat{Y} \\ \hat{r} \end{pmatrix} = \frac{1}{g} \begin{pmatrix} f & d/P \\ e & -(1-b)/P \end{pmatrix} \begin{pmatrix} a + c + \hat{G} - b\hat{T} \\ M \end{pmatrix},$$

by linearity, we have

$$\begin{pmatrix} \hat{Y} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} Y \\ r \end{pmatrix} + (\delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G) - b\delta_{\mathfrak{a}}(T) - b\delta_{\mathfrak{d}}(T)) \frac{1}{g} \begin{pmatrix} f \\ e \end{pmatrix}.$$

The use of Game Theory gives the opportunity to study this situation under different "intermediate" scenarios with a larger number of possibilities to consider. We adopted in Theorem 9 a moderate view respecting to goodness and badness: there is too much luck in assuming that measures over G and T will be successful all together. This translates in the rule: it is possible to apply $\delta_{\mathfrak{a}}(G)$ or $\delta_{\mathfrak{a}}(T)$ but not both *at the same time*. Our point about badness is similar, things are bad but not too bad, translating into similar rules for \mathfrak{d} 's actions. We are not saying that it is not possible to simultaneously change government spending and taxation. But if this is the case, the design of the uncertainty profile, and in consequence of the $\mathfrak{a}/\mathfrak{d}$ game, will be different.

Let us comment Theorem 9. The possible results depends on $\mu_{\mathfrak{p}, T \rightarrow G}$ for $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$. Let us consider some cases. When $\mu_{\mathfrak{p}, T \rightarrow G} > 0$ for $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$ the strategy profile $(\{G\}\{G\})$ is the unique PNE with utilities

$$Y(\{G\}\{G\}) = Y + \frac{f}{g}(\delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G)), r(\{G\}\{G\}) = Y + \frac{e}{g}(\delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G)).$$

In such a case, just government spending is a good policy rule. Note that perturbations exerted to taxes $(\delta_{\mathfrak{a}}(T), \delta_{\mathfrak{d}}(T))$ do not appear). Therefore, it seems better to keep taxes unchanged. The relation between both approaches is given by

$$\begin{pmatrix} \hat{Y} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} Y(\{G\}\{G\}) \\ r(\{G\}\{G\}) \end{pmatrix} - b(\delta_{\mathfrak{a}}(T) + \delta_{\mathfrak{d}}(T)) \frac{1}{g} \begin{pmatrix} f \\ e \end{pmatrix}.$$

Thus, depending on the sign of $\delta_{\mathfrak{a}}(T) + \delta_{\mathfrak{d}}(T)$, one solution or the other one will be the best one. When $\mu_{\mathfrak{p}, T \rightarrow G} < 0$, for $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{d}\}$, the strategy profile $(\{T\}\{T\})$ is the unique PNE. In such a case taxing is the good solution and the relation between both approaches is given by

$$\begin{pmatrix} \hat{Y} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} Y(\{T\}\{T\}) \\ r(\{T\}\{T\}) \end{pmatrix} + (\delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G)) \frac{1}{g} \begin{pmatrix} f \\ e \end{pmatrix}.$$

An interesting case happens when $\mu_{\mathfrak{a}, T \rightarrow G} > 0$ and $\mu_{\mathfrak{d}, T \rightarrow G} < 0$. In this situation $\{T, G\}$ is the unique PNE with values

$$Y(\{T\}\{G\}) = Y + \frac{f}{g}(\delta_{\mathfrak{d}}(G) - b\delta_{\mathfrak{a}}(T)), r(\{T\}\{G\}) = Y + \frac{e}{g}(\delta_{\mathfrak{d}}(G) - b\delta_{\mathfrak{a}}(T)).$$

Thus highlighting a case where both, government spending and taxing, will occur in equilibrium. Comparing with \hat{Y} and \hat{r} we have that

$$\begin{pmatrix} \hat{Y} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} Y(\{T\}\{T\}) \\ r(\{T\}\{T\}) \end{pmatrix} + (\delta_{\mathfrak{a}}(G) - b\delta_{\mathfrak{d}}(T)) \frac{1}{g} \begin{pmatrix} f \\ e \end{pmatrix}.$$

Uncertainty profiles help us to catch and deal with many intermediate cases between total successfulness and complete failure. Note that uncertainty profiles are flexible tools. For instance we can catch the solution \hat{Y} and \hat{r} with an uncertainty profile as we show in the following example

Example 6.1. Take a perturbation strength model \mathcal{S} as in Theorem 9 and consider the uncertainty profile $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \{T, G\}, \{T, G\}, 2, 2, Y, r \rangle$. Lemma 6 deals with this case having $(\{T, G\}, \{T, G\})$ as a unique PNE. The corresponding utilities are \hat{Y} and \hat{r} . \square

6.2 On Balanced Budgets

Now we consider an $\mathfrak{a}/\mathfrak{d}$ approach to deal with balanced budgets. To do that, we take a special case of the perturbation strength model \mathcal{S} considered in Theorem 9 where the government spends the amount δ collected by taxation. We call it $\mathcal{S}_{balanced}$,

agent	a, b, c, d, e, f	T	G	M, P
\mathfrak{a}	0	0	δ	0
\mathfrak{d}	0	δ	0	0

Consider the uncertainty profile $\mathcal{U}_{balanced} = \langle \mathcal{E}, \mathcal{S}_{balanced}, \{G, T\}, \{G, T\}, 1, 1, Y, -r \rangle$. From Theorem 9, we easily get the following description for $\Gamma(\mathcal{U}_{balanced})$

		\mathfrak{d}	
		$\{T\}$	$\{G\}$
\mathfrak{a}	$\{T\}$	$Y - \frac{f}{g}b\delta, -(r - \frac{e}{g}b\delta)$	$Y, -r$
	$\{G\}$	$Y + \frac{f}{g}\delta(1 - b), -(r + \frac{e}{g}\delta(1 - \delta))$	$Y + \frac{f}{g}\delta, -(r + \frac{e}{g}\delta)$

The sets of best responses are $B_{\mathfrak{a}}(\{T\}) = B_{\mathfrak{a}}(\{G\}) = \{G\}$ and $B_{\mathfrak{d}}(\{T\}) = B_{\mathfrak{d}}(\{G\}) = \{T\}$. The only PNE is $(\{G\}, \{T\})$. This Nash equilibrium corresponds to the case where Government spends the amount δ collected by taxes and this corresponds to a balanced budget.

6.3 Direct Control over Income and Interest Rate from \mathfrak{a} and \mathfrak{d}

In various previous examples giving raise to non zero-sum games we have considered $u_{\mathfrak{a}} = Y$ and $u_{\mathfrak{d}} = r$. In such cases, the goal of \mathfrak{a} consists on maximize the Y (a good thing) while \mathfrak{d} tries to maximize the interest (a bad thing). We consider now the case where \mathfrak{a} is interested in controlling both, Y and r , directly. Informally \mathfrak{a} likes to keep Y as high as possible and in order to get that r needs to be as small as possible. A way to model this situation is to consider $u_{\mathfrak{a}} = Y - r$. Now \mathfrak{a} is interested in increasing Y and decreasing r in order to keep $u_{\mathfrak{a}}$ as high as possible. We can tune the relevance of Y in front of r using a parameter $k > 0$ and defining $u_{\mathfrak{a}}(a, d) = Y(a, d) - k r(a, d)$. In the case that \mathfrak{d} is interested just in the opposite, $u_{\mathfrak{d}}(a, d) = k r(a, d) - Y(a, d)$. Therefore, $u_{\mathfrak{a}} = -u_{\mathfrak{d}}$ and $\Gamma(\mathcal{U})$ is a zero sum game with $u(a, d) = u_{\mathfrak{a}}(a, d)$. Our next results analyzes the PNE of a case of perturbed fiscal policy where any effort undertaken by \mathfrak{a} can be cancelled by \mathfrak{d} and vice-versa, this can be modeled by letting $\delta_{\mathfrak{a}}(T) + \delta_{\mathfrak{d}}(T) = 0$ and $\delta_{\mathfrak{a}}(G) + \delta_{\mathfrak{d}}(G) = 0$.

Theorem 10. *Let \mathcal{S} be a perturbation strength model for the IS-LM model given by*

agent	a, b, c, d, e, f	T	G	M, P
\mathfrak{a}	0	$\delta_{\mathfrak{a}}(T) = -\delta_T$	$\delta_{\mathfrak{a}}(G) = \delta_G$	0
\mathfrak{d}	0	$\delta_{\mathfrak{d}}(T) = \delta_T$	$\delta_{\mathfrak{d}}(G) = -\delta_G$	0

Consider the uncertainty profile $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \{T, G\}, \{T, G\}, 1, 1, u \rangle$ where $u(a, d) = Y(a, d) - k r(a, d)$, for some $\delta_T > 0$ and $\delta_G \geq 0$. Letting $\delta = b\delta_T - \delta_G$, the associated zero-sum $\mathfrak{a}/\mathfrak{d}$ game $\Gamma(\mathcal{U})$ is given by

		\mathfrak{d}	
		$\{T\}$	$\{G\}$
\mathfrak{a}	$\{T\}$	$Y - k r$	$(Y - k r) + (1/g)(f - k e)\delta$
	$\{G\}$	$(Y - k r) - (1/g)(f - k e)\delta$	$Y - k r$

When $(f - k e)\delta = 0$, the four strategy profiles of $\Gamma(\mathcal{U})$ are PNE, otherwise either $(\{T\}, \{T\})$ or $(\{G\}, \{G\})$ is the unique PNE of $\Gamma(\mathcal{U})$.

Proof. Let us compute the utilities in $\Gamma(\mathcal{U})$. We can easily get $Y(\{T\}, \{T\}) = Y(\{G\}, \{G\}) = Y$ and $r(\{T\}, \{T\}) = r(\{G\}, \{G\}) = r$. Recall that the expression of the equilibrium point is,

$$Y(\{T\}, \{G\}) = \frac{f}{g} \left(a + c + G - \delta_G - b(T - \delta_T) \right) + \frac{d}{g} \frac{M}{P} = Y + \frac{f}{g} \delta.$$

Thus, summarizing we have

$$\begin{aligned} Y(\{T\}, \{G\}) &= Y + (f/g)\delta, \quad r(\{T\}, \{G\}) = r + (e/g)\delta, \\ Y(\{G\}, \{T\}) &= Y - (f/g)\delta, \quad r(\{G\}, \{T\}) = r - (e/g)\delta. \end{aligned}$$

Considering the utilities, clearly $u(\{T\}, \{T\}) = u(\{G\}, \{G\}) = Y - kr$ and

$$\begin{aligned} u(\{T\}, \{G\}) &= Y(\{T\}, \{G\}) - kr(\{T\}, \{G\}) \\ &= Y + (f/g)\delta - k(r + (e/g)\delta) = Y - kr + (1/g)(f - ke)\delta. \end{aligned}$$

In a similar way $u(\{G\}, \{T\}) = (Y - kr) - (1/g)(f - ke)\delta$ and we get the claimed expression for $\Gamma(\mathcal{U})$. Let us analyze the structure of the PNE. When $(f - ke)\delta = 0$ all the pure strategies have the same utility $Y - kr$ and therefore all of them are PNE. When $Y - kr > 0$ and $(f - ke)\delta > 0$ the only PNE is $(\{T\}, \{T\})$. When $Y - kr > 0$ and $(f - ke)\delta < 0$ the only PNE is $(\{G\}, \{G\})$. When $Y - kr < 0$ and $(f - ke)\delta > 0$ the only PNE is $(\{T\}, \{T\})$. When $Y - kr < 0$ and $(f - ke)\delta < 0$ the only PNE is $(\{G\}, \{G\})$.

When $k = Y/r$, reconsidering Example 5.1 taking $u(a, d) = Y(a, d) - (Y/r)r(a, d)$ and Theorem 10, we have that $\Gamma(\mathcal{U})$ is

		\mathfrak{d}	
		$\{T\}$	$\{G\}$
\mathfrak{a}	$\{T\}$	0	$(1/g)(f - (Y/r)e)\delta$
	$\{G\}$	$-(1/g)(f - (Y/r)e)\delta$	0

When $u(\{T\}, \{G\}) > 0$, the game has a PNE in $(\{T\}, \{T\})$. When $u(\{T\}, \{G\}) < 0$ the PNE is $(\{G\}, \{G\})$. This concludes the proof. \square

7 Uncertainty in the IS-MP Model

Now we consider the case where the income Y and the inflation π become uncertain due to perturbations in $\{\bar{Y}, \pi^*\}$. We analyse the case where a benevolent \mathfrak{a} tries to keep the income as high as possible, $u_{\mathfrak{a}} = Y$. As in Example 5.2, we consider different views of \mathfrak{d} in relation to r . To model the case where \mathfrak{d} tries to maximize the inflation, we take $u_{\mathfrak{d}} = \pi$. To model the case where π is a dis-utility \mathfrak{d} tries to minimize π , we take $u_{\mathfrak{d}} = -\pi$. The following theorem consider both cases.

Theorem 11. *Let S be a perturbation strength model for the IS-MP model with $\delta_{\mathfrak{a}}(e) = \delta_{\mathfrak{d}}(e) = 0$, for any $e \in \mathcal{E} \setminus \{\bar{Y}, \pi^*\}$ and $\delta_{\mathfrak{a}}(\bar{Y}) > 0$ and $\delta_{\mathfrak{d}}(\pi^*) > 0$. Consider and uncertainty profile $\mathcal{U} = \langle \mathcal{E}, S, \{\bar{Y}, \pi^*\}, \{\bar{Y}, \pi^*\}, 1, 1, Y, u_{\mathfrak{d}} \rangle$. When $u_{\mathfrak{d}} = \pi$ the $\mathfrak{a}/\mathfrak{d}$ game is*

		\mathfrak{d}	
		$\{\bar{Y}\}$	$\{\pi^*\}$
\mathfrak{a}	$\{\bar{Y}\}$	$Y + \delta_{\mathfrak{a}}(\bar{Y}) + \delta_{\mathfrak{d}}(\bar{Y}), \pi$	$Y + \delta_{\mathfrak{a}}(\bar{Y}), \pi + \delta_{\mathfrak{d}}(\pi^*)$
	$\{\pi^*\}$	$Y + \delta_{\mathfrak{d}}(\bar{Y}), \pi + \delta_{\mathfrak{a}}(\pi^*)$	$Y, \pi + \delta_{\mathfrak{a}}(\pi^*) + \delta_{\mathfrak{d}}(\pi^*)$

$\{\bar{Y}\}$ is the dominant strategy for \mathfrak{a} and $\{\pi^*\}$ is the dominant strategy for \mathfrak{d} . The unique PNE is $(\{\bar{Y}\}, \{\pi^*\})$ with $Y(\{\bar{Y}\}, \{\pi^*\}) = Y + \delta_{\mathfrak{a}}(\bar{Y})$ and $\pi(\{\bar{Y}\}, \{\pi^*\}) = \pi + \delta_{\mathfrak{a}}(\pi^*)$. When $u_{\mathfrak{d}} = -\pi$ the game is

		\mathfrak{d}	
		$\{\bar{Y}\}$	$\{\pi^*\}$
\mathfrak{a}	$\{\bar{Y}\}$	$Y + \delta_{\mathfrak{a}}(\bar{Y}) + \delta_{\mathfrak{d}}(\bar{Y}), -\pi$	$Y + \delta_{\mathfrak{a}}(\bar{Y}), -\pi + \delta_{\mathfrak{d}}(\pi^*)$
	$\{\pi^*\}$	$Y + \delta_{\mathfrak{d}}(\bar{Y}), -\pi + \delta_{\mathfrak{a}}(\pi^*)$	$Y, -\pi + \delta_{\mathfrak{a}}(\pi^*) + \delta_{\mathfrak{d}}(\pi^*)$

The dominant strategy for both \mathfrak{a} and \mathfrak{d} is $\{\bar{Y}\}$ and the only PNE is $(\{\bar{Y}\}, \{\bar{Y}\})$. In this case $Y(\{\bar{Y}\}, \{\bar{Y}\}) = Y + \delta_{\mathfrak{a}}(\bar{Y}) + \delta_{\mathfrak{d}}(\bar{Y})$ and $\pi(\{\bar{Y}\}, \{\bar{Y}\}) = \pi$.

Proof. Let us consider the game $\Gamma(\mathcal{U})$. According to Lemma 5 we have

$$u_{\mathfrak{a}}(a, d) = Y(a, d) = Y + \delta_{\mathcal{S}}(\bar{Y})[a, d], \quad u_{\mathfrak{d}}(a, d) = \pi(a, d) = \pi + \delta_{\mathcal{S}}(\pi^*)[a, d]$$

The set of strategy profiles is:

$$A_{\mathfrak{a}} \times A_{\mathfrak{d}} = \{(\{\bar{Y}\}, \{\bar{Y}\}), (\{\bar{Y}\}, \{\pi^*\}), (\{\pi^*\}, \{\bar{Y}\}), (\{\pi^*\}, \{\pi^*\})\}$$

This gives the following bi-matrix form.

		\mathfrak{d}	
		$\{\bar{Y}\}$	$\{\pi^*\}$
\mathfrak{a}	$\{\bar{Y}\}$	$Y + \delta_{\mathcal{S}}(\bar{Y})[\{\bar{Y}\}, \{\bar{Y}\}], \pi + \delta_{\mathcal{S}}(\pi^*)[\{\bar{Y}\}, \{\bar{Y}\}]$	\dots, \dots
	$\{\pi^*\}$	$Y + \delta_{\mathcal{S}}(\bar{Y})[\{\pi^*\}, \{\bar{Y}\}], \pi + \delta_{\mathcal{S}}(\pi^*)[\{\pi^*\}, \{\bar{Y}\}]$	\dots, \dots

Consider the strategy profile $(a, d) = (\{\bar{Y}\}, \{\bar{Y}\})$. According to the Definition 4, as $\bar{Y} \in a \cap d = \{\bar{Y}\}$, it holds $\delta_{\mathcal{S}}(\bar{Y})[a, d] = \delta_{\mathfrak{a}}(\bar{Y}) + \delta_{\mathfrak{d}}(\bar{Y})$ and $u_{\mathfrak{a}}(\{\bar{Y}\}, \{\bar{Y}\}) = Y + \delta_{\mathfrak{a}}(\bar{Y}) + \delta_{\mathfrak{d}}(\bar{Y})$. As $\pi \notin a \cup d = \{\bar{Y}\}$, the perturbation strength verifies $\delta_{\mathcal{S}}(\pi^*)[a, d] = 0$ and therefore $u_{\mathfrak{d}}(\{\bar{Y}\}, \{\bar{Y}\}) = \pi$. The remaining cases are similar and thus we get the claimed bi-matrix form for $\Gamma(\mathcal{U})$. Let us analyse dominance. For \mathfrak{a} , as $\delta_{\mathfrak{a}}(\bar{Y}) > 0$, independently of the sign of $\delta_{\mathfrak{d}}(\pi^*)$, best responses verify $B_{\mathfrak{a}}(\bar{Y}) = B_{\mathfrak{a}}(\pi^*) = \bar{Y}$. For \mathfrak{d} , as $\delta_{\mathfrak{d}}(\pi^*) > 0$, best responses are $B_{\mathfrak{d}}(\bar{Y}) = B_{\mathfrak{d}}(\pi^*) = \pi^*$. The proof for \mathcal{U}_2 is similar and the claim follows. \square

It is worth observing that for $\langle \mathcal{E}, \mathcal{S}, \{\bar{Y}, \pi^*\}, \{\bar{Y}, \pi^*\}, 1, 1, Y, \pi \rangle$, the selfish behaviour of \mathfrak{a} and \mathfrak{d} give a PNE $(\{\bar{Y}\}, \{\pi^*\})$ corresponding to a "moderate" perturbed values i.e., $Y(\{\bar{Y}\}, \{\pi^*\}) = Y + \delta_{\mathfrak{a}}(\bar{Y})$ and $\pi(\{\bar{Y}\}, \{\pi^*\}) = \pi + \delta_{\mathfrak{d}}(\pi^*)$. Selfish behaviour protect us from extreme behaviours with maximal income or maximal inflation. The two extreme cases have utilities $(Y + \delta_{\mathfrak{a}}(\bar{Y}) + \delta_{\mathfrak{d}}(\bar{Y}), \pi)$ and $(Y, \pi + \delta_{\mathfrak{a}}(\pi^*) + \delta_{\mathfrak{d}}(\pi^*))$. Situation changes when $\langle \mathcal{E}, \mathcal{S}, \{\bar{Y}, \pi^*\}, \{\bar{Y}, \pi^*\}, 1, 1, Y, -\pi \rangle$. In this case situation evolves to a PNE $(\{\bar{Y}\}, \{\bar{Y}\})$ such that $Y(\{\bar{Y}\}, \{\bar{Y}\}) = Y + \delta_{\mathfrak{a}}(\bar{Y}) + \delta_{\mathfrak{d}}(\bar{Y})$ and $\pi(\{\bar{Y}\}, \{\bar{Y}\}) = \pi$. In such a case \mathfrak{a} and \mathfrak{d} move on to get high income Y and to do not touch at the inflation π .

A possible practical implication of the Theorem 7 follows. The perturbation model \mathcal{S} acts over \bar{Y} and π^* through $\delta_{\mathfrak{a}}(\bar{Y}) > 0$, $\delta_{\mathfrak{a}}(\pi^*)$, $\delta_{\mathfrak{d}}(\bar{Y})$, $\delta_{\mathfrak{d}}(\pi^*) > 0$. When $u_{\mathfrak{d}} = \pi$ the PNE is $(\{\bar{Y}\}, \{\pi^*\})$, when $u_{\mathfrak{d}} = -\pi$ the PNE is $(\{\bar{Y}\}, \{\bar{Y}\})$ it is worth noting that the Nash equilibria are independent of the signs of $\delta_{\mathfrak{a}}(\pi^*)$ and $\delta_{\mathfrak{d}}(\bar{Y})$. As a consequence any policy trying to act over $\delta_{\mathfrak{a}}(\pi^*)$ or $\delta_{\mathfrak{d}}(\bar{Y})$ will not modify the general structure of the Nash equilibria.

We conclude our study considering a subclass of zero-sum $\mathfrak{a}/\mathfrak{d}$ games generalizing the conditions of the uncertainty profiles considered in Example 5.2.

Theorem 12. Let S be a perturbation strength model for the IS-MP model such that $\delta_a(e) = \delta_d(e) = 0$, for any $e \in \mathcal{P}$. Let $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, 1, 1, u \rangle$ where

$$u(a, d) = Y(a, d) - (Y(\{\emptyset\}, \{\emptyset\})/\pi(\{\emptyset\}, \{\emptyset\}))\pi(a, d).$$

The associated zero-sum \mathbf{a}/\mathbf{d} game $\Gamma(\mathcal{U})$ is:

		\mathbf{d}	
		$\{\bar{Y}\}$	$\{\pi^*\}$
\mathbf{a}	$\{\bar{Y}\}$	$\delta_a(\bar{Y}) + \delta_d(\bar{Y})$	$\delta_a(\bar{Y}) - (Y/M)\delta_d(\pi^*)$
	$\{\pi^*\}$	$\delta_d(\bar{Y}) - (Y/\pi)\delta_a(\pi^*)$	$-(Y/\pi)(\delta_a(\pi^*) + \delta_d(\pi^*))$

When $\delta_p(e) > 0$, for $\mathbf{p} \in \{\mathbf{a}, \mathbf{d}\}$ and $e \in \{\bar{Y}, \pi^*\}$, $\{\bar{Y}\}$ is the dominant strategy for \mathbf{a} and $\{\pi^*\}$ is the dominant strategy for \mathbf{d} . Furthermore, $(\{\bar{Y}\}, \{\pi^*\})$ is the unique PNE.

Proof. The tabular form of $\Gamma(\mathcal{U})$ comes straightforward from the tabular form appearing in the Theorem 11. With respect to dominance, let us compute best responses. The \mathbf{a} best response to $\{\bar{Y}\}$, as $\delta_a(\bar{Y}) + \delta_d(\bar{Y}) > \delta_d(\bar{Y}) - (Y/\pi)\delta_a(\pi^*)$, verifies $B_a(\{\bar{Y}\}) = \{\bar{Y}\}$ and similarly $B_a(\{\pi^*\}) = \{\bar{Y}\}$. Consider now the best responses of \mathbf{d} . When \mathbf{a} chooses $\{\bar{Y}\}$, as $\delta_a(\bar{Y}) + \delta_d(\bar{Y}) > \delta_a(\bar{Y}) - (Y/M)\delta_d(\pi^*)$ and \mathbf{d} tries to minimize the utility, we get $B_d(\{\bar{Y}\}) = \{\pi^*\}$. Similarly, when \mathbf{a} chooses $\{\pi^*\}$ we get $B_d(\{\pi^*\}) = \{\pi^*\}$. \square

8 Conclusions and Further Developments

We have shown how to adapt the \mathbf{a}/\mathbf{d} -framework provided in (Gabarro et al., 2014) to analyse uncertainty in the IS-LM and the IS-MP models. In both cases, we have studied different possible cases of uncertainty through the set of Nash equilibria showing the applicability of the framework. In general, a common way to deal with uncertainty consists on *modelling several scenarios* and develop qualitative likelihood analysis of the different cases. In our proposed analysis positive and negative aspects are taken into account.

Our approach can be adapted to analyse uncertainty in other financial settings. As an example, consider an option valuation using the Greeks (Hull, 1989). We develop now a possible application based on Δ where Δ is the ratio between the changes in the price of the derivative to the corresponding price of the underlying asset. We do that considering uncertainty in the price f of a call option over a stock S . We use the one step binomial tree model (Hull, 1989) having the following components:

Stock price	S	Time period	T	Up jump	u
Strike price	X	Risk-free rate	r	Down jump	d

Where S is the price of the stock, X is the strike price (the price at the expiration date), T is the time period before expiration. The parameters u and d fixed such that, if stock goes up the price (of the stock) should be $S \times u$ and if it goes down the price should be $S \times d$. Consider the following valuation taken from (Hull, 1989) serving us to introduce the basic constructs:

S	X	T	r	u	d
20	21	0.25	0.12	1.1	0.6

In order to compute f , consider a portfolio having a long Δ shares and one short call option. If the stock goes up in 3 months ($T = 0.25$) it holds $S \times u > X$ and the payoff from the derivative is $f_u = S \times u - X = 1$. The value of the portfolio in the up case is $S \times u \times \Delta - f_u$. If the stock goes down, $f_d = 0$ the value is $S \times d \times \Delta - f_d$. In order to be risk-free Δ needs to verify $S \times u \times \Delta - f_u = S \times d \times \Delta - f_d$. Therefore

$$\Delta = \frac{f_u - f_d}{S \times u - S \times d} = \frac{1}{10}$$

and the value of the portfolio at the end of the period will be $S \times d \times \Delta$. To get the present value v we discount at free-risk rate $v = S \times d \times \Delta \times e^{-r \times T} = 1.164535$ and the price f of the call is $f = S \times \Delta \times (1 - v) = S \times \Delta \times (1 - d \times e^{-r \times T}) = 0.835465$. As in a call the values S , X and T are explicitly is written, therefore is no ambiguity about them. Therefore the set of exogenous components is $\mathcal{E} = \{r, u, d\}$. An example of a perturbation strength model \mathcal{S} is

p	r	u	d
a	-0.05	+0.4	0
d	+0.10	0	+0.3

Given the strategy profile (a, d) the new values of the perturbed set are $\mathcal{E}' = \{r', u', d'\}$. Let, consider the case $(a, d) = (\{u\}, \{d\})$. As $r \notin \{u, d\}$ it holds is $r' = r$. As u is only selected by the angel, $u' = u + \delta_a(u) = 1.1 + 0.4 = 1.5$. As d is just selected by the daemon $d' = d + \delta_d(d) = 0.6 + 0.3 = 0.9$. As $S \times u' = 30$ and $f_{u'} = 30 - 21 = 9$. As $S \times d' = 18$ we have $f_{d'} = 0$. As usual we shorten $\Delta(\text{strength}_{\mathcal{S}}[\{u\}, \{d\}])$ as $\Delta(\{u\}, \{d\}) = 3/4$ and $f(\{u\}, \{d\}) = S \times \Delta(\{u\}, \{d\}) \times (1 - d' \times e^{-r' \times T}) = 1.898985$. As another case example of interest consider the strategy where both a and d choose to strength r . In this case that is $(a, d) = (\{r\}, \{r\})$. In this case $u' = u = 1.1$, $d' = d = 0.6$ and $r' = r + \delta_a(r) + \delta_d(r) = 0.12 - 0.05 + 0.10 = 0.17$. As neither u nor d has been perturbed, $\Delta(\{r\}, \{r\}) = \Delta = 1/10$ and $f(\{r\}, \{r\}) = 0.8499314$. Let $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \{\{u\}, \{r\}\}, \{\{d\}, \{r\}\}, 1, 1, f \rangle$ be an uncertainty profile for a call option. The corresponding a/d game is:

		d	
		$\{d\}$	$\{r\}$
a	$\{u\}$	1.898985	4.321089
	$\{r\}$	0.5780649	0.8499314

The only PNE is $(\{u\}, \{d\})$ with $\nu(\mathcal{U}) = 1.898985$. Therefore, in the uncertain situation described by \mathcal{U} perhaps the price of the call should be 1.898985 rather than 0.835465.

Studying the applicability of the a/d approach in other macroeconomic settings is of great interest. In particular we are interested in compare our approach to other provided analytical tools. To do so we would need to further analyse the structure and properties of the Nash equilibria of the associated games. It is important to know the possibilities and limits of the uncertainty profiles and a/d games in macroeconomics. As pointed in (Durlauf, 2012), a way to decide with no probabilities is to guard against really bad cases. In such a setting regret analysis provides a way to analyse such cases. Analysing the connection between a/d and regret analysis would be interesting. In (Nordhaus, 2013), different scenarios are developed in order to analyse different possibilities in relation to climate change. For instance, when considering the cost of meeting global temperature targets, two possible scenarios for

the average-cost/temperature are considered. Those models are not far away from the ones considered in this paper, thus uncertainty profiles and α/\mathfrak{d} could provide a complementary analysis tool. Varian (1977) has studied the stability of the IS-LM model. As $(Y(a, d), r(a, d))$ can be seen as perturbations of the equilibrium point (Y, r) (see Lemmas 2, 3) it will be of interest to study the relationship among of both models. In general, it would be interesting to try to apply α/\mathfrak{d} -framework to non-linear models. Even if general theorems seems difficult to obtain, accurate numerical examples could be a first step to perform the comparison. Moreover, in order to transform α/\mathfrak{d} -framework into a practical tool it seems necessary to explore and develop the links with macroeconomic policy coordination and international macroeconomic integration.

It is also important to know the possibilities, limits and weaknesses of the α/\mathfrak{d} approach. We have considered uncertainty profiles $\mathcal{U} = \langle \mathcal{E}, \mathcal{S}, \mathcal{A}, \mathcal{D}, b_a, b_d, u_a, u_d \rangle$ where the spreads b_a, b_d are fixed independently of the other parameters in \mathcal{U} . It is interesting to consider the case where the spreads are related to other parameters (see the first part of the Lemma 6), for instance $b_a = \lfloor \frac{1}{2} \# \mathcal{A} \rfloor$ or $b_d = \lfloor \frac{2}{3} \# \mathcal{A} \rfloor$. Moreover, any \mathcal{U} is a tuple having many parameters to be determined. In principle those parameters are fixed by the analyser based on his perception of the world and historical data. However the danger of over parametrization exists (Hull, 2010). Remind the von Neumann dictum, “with four parameters I can fit an elephant and with five I can make him wiggle his trunk” (Dyson, 2004). Also possible connections between uncertainty profiles and α/\mathfrak{d} games and the maxmin expected utility approach proposed by (Gilboa & Schmeidler, 1989) and developed in (Hansen et al., 2006a,b) deserves further study.

Finally, in this paper we have considered only short-time models. The α/\mathfrak{d} approach for long-time models is left open. A common way to deal with uncertainty along the time consists on *modelling the temporal evolution* of several possible scenarios. As an example, consider the two possible scenarios in the evolution (France, 1820-2100) of the annual value of bequest and gifts (Piketty, 2014). Scenarios are given in terms of the growth of output g and of the rate of return on capital r (do not confuse with the previous interest rate r). There is a *central scenario* with $g = 1.7\%$ and $r = 3.0\%$ and an *alternative scenario* with $g = 1.0\%$ and $r = 5.0\%$. In order to tackle such cases, temporal aspects can be incorporated to game theory using stochastic games (Shapley, 1953). In (Castro et al., 2015) a time dimension of the α/\mathfrak{d} approach was integrated into the frame of stochastic automata. Perhaps this approach, based on stochastic automata, could be adapted to deal with the uncertainty along the time of some economic models.

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